SCIENTIFIC PUBLICATIONS OF THE STATE UNIVERSITY OF NOVI PAZAR
SER. A: APPL. MATH. INFORM. AND MECH. vol. 6, 2 (2014), 119-129.

Trees Whose Second Largest Eigenvalue Does Not Exceed $\frac{\sqrt{5}+1}{2}$

B. Lj. Mihailović

Abstract: The second largest eigenvalue (λ_2) provides significant information on characteristics and structure of graphs. Therefore, finding bounds for λ_2 is a topic of interest in many fields. In this paper we prove one general theorem about values of λ_2 of graphs with a cut-vertex and after that we determine all trees with the property $\lambda_2 \leq \frac{1+\sqrt{5}}{2}$.
Keywords: spectral graph theory, tree, second largest eigenvalue

1 Introduction

In this paper we consider connected simple graphs, i.e. undirected, with no loops or multiple edges. When we use the term *subgraph* (*subtree*), it means the *induced subgraph* (*subtree*). Naturally, H is a supergraph of G if G is a subgraph (induced!) of H. If A is $(0,1)$-adjacency matrix of a graph G, then its *characteristic polynomial* is defined by $P_G(\lambda) = det(\lambda I - A)$. The roots of the characteristic polynomial are called the *eigenvalues* of G. The family of eigenvalues is the *spectrum* of G. A is a real and symmetric matrix, and, therefore, its eigenvalues are real. We assume their non-increasing order: $\lambda_1(G) \geq \lambda_2(G) \geq \ldots \geq \lambda_n(G)$. The largest eigenvalue $\lambda_1(G)$ is called the *index* of G. For connected graphs $\lambda_1(G) > \lambda_2(G)$ holds. If graph G is not connected, then its spectrum is the union of the spectra of its components.

The second largest eigenvalue of a graph is a subject of investigations in spectral graph theory, but also in computer science and various fields across the science in which networks as mathematical models are widely used.

In spectral graph theory, graphs with the second largest eigenvalue bounded by a constant $a \in \mathbb{R}$ have been investigated by many authors. Some of the bounds considered so far are: $a = \frac{1}{3}$ [1], $a = \sqrt{2} - 1$ [6], $a = \frac{\sqrt{5}-1}{2}$ [3], $a = 1$ [4], $a = \sqrt{2}$ [5], $a = \sqrt{3}$ [5]. Reflexive graphs ($a = 2$) have been investigated in many articles, for example [8, 9, 10, 11, 12] where

Manuscript received April 11 ; accepted June 30
B. Lj. Mihailović is with the Faculty of Electrical Engineering, University of Belgrade, Serbia, e-mail mihailovicb@etf.rs

Theorem 1 of [10] (further, RS-theorem) was often used to prove whether a connected graph with a cut-vertex is reflexive or not.

In this paper we shall prove a generalization of this theorem and then use it in determining all trees whose second largest eigenvalue does not exceed $\frac{\sqrt{5}+1}{2}$. This bound for the second largest eigenvalue has not been considered before.

The paper is structured in the following way. In Section 2, we present the main tools used in our investigations including the RS-theorem. Section 3 brings the Generalized RS-theorem, along with two auxiliary lemmas. In section 4, applying this theorem and by further analysis of remaining cases, we determine all trees whose second largest eigenvalue does not exceed $\frac{\sqrt{5}+1}{2}$.

2 Auxiliary results and the RS-theorem

The following theorem shows the interrelation between the spectra of a graph and its induced subgraph.

The Interlacing Theorem (e.g.[2]) Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of a simple graph G and $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$ the eigenvalues of its induced subgraph H. Then the inequalities $\lambda_{n-m+i} \leq \mu_i \leq \lambda_i$ $(i = 1, 2, \ldots, m)$ hold.

If G is a connected graph and $m = n-1$, then $\lambda_1 > \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots$.

Schwenk's Lemma (e.g.[6]) Given a graph G, let $C(v)$ and $C(uv)$ denote the set of all cycles containing a vertex v and an edge uv of G, respectively. Then

1. $P_G(\lambda) = \lambda P_{G-v}(\lambda) - \sum_{u \in Adj(v)} P_{G-v-u}(\lambda) - 2\sum_{C \in C(v)} P_{G-V(C)}(\lambda)$,
2. $P_G(\lambda) = P_{G-uv}(\lambda) -_{G-v-u}(\lambda) - 2\sum_{C \in C(uv)} P_{G-V(C)}(\lambda)$,

where $Adj(v)$ denotes the set of neighbours of v, while $G - V(C)$ is the graph obtained from G by removing the vertices belonging to the cycle C.

The only connected graphs for which $\lambda_1 = 2$ holds are called Smith graphs [13]. For every connected graph exactly one of the following statements hold: 1) a graph is a Smith graph, 2) a graph is a proper subgraph of some Smith graphs and its index is less than 2, 3) a graph is a proper supergraph of some of Smith graphs and its index is greater than 2.

It can be established effortlessly in many cases whether a graph with a cut-vertex is reflexive or not using a theorem of [10] (Z. Radosavljevic, S. Simic) which for convenience we will call RS-theorem.

RS-theorem [10] Let G be a graph with a cut-vertex u.

1. If at least two components of $G-u$ are supergraphs of Smith graphs, and if at least one of them is a proper supergraph, then $\lambda_2(G) > 2$.
2. If at least two components of $G-u$ are Smith graphs, and the rest are subgraphs of Smith graphs, then $\lambda_2(G) = 2$.

3. If at most one component of $G-u$ is a Smith graph, and the rest are proper subgraphs of Smith graphs, then $\lambda_2(G) < 2$.

After removing a cut-vertex of the graph G, we get several new connected graphs. They are comparable to the Smith graphs, in the sense that they are either their subgraphs or supergraphs.

In many cases the theorem gives the answer about the reflexivity of the graph, but there is one case when it does not, namely when after the removal of the cut-vertex u we get one proper supergraph, and the rest are proper subgraphs of Smith graphs. In these cases other techniques are used, but often, at the subgraph level analysis, the RS-theorem proves useful again.

In former work, many classes of reflexive graphs have been described. Since reflexivity is a hereditary property, i.e. every subgraph preserves it, it is natural to present classes of resulting graphs through the sets of maximal reflexive graphs. In this case maximal means that its supergraphs are not reflexive. It turned out that Smith graphs play an essential role also in the construction of maximal reflexive graphs [7, 8, 9, 11, 12] By generalizing the RS-theorem we get a useful instrument for comparing λ_2 with arbitrary $a > 0$ for many graphs with a cut-vertex.

3 The Generalized RS-theorem

In this section we will prove the Generalized RS-theorem. Before that, we present two lemmas that will be used in the proof of the theorem.

Lemma 3.1 *Consider graph G in Fig. 1, where G_1 is a connected graph with the index a, $a > 0$, and u is an extra vertex connected to some of the vertices of the graph G_1 $(v_1, v_2, \ldots, v_m)$. Then, $P_G(a) < 0$, and, consequently, $\lambda_2(G) < a < \lambda_1(G)$.*

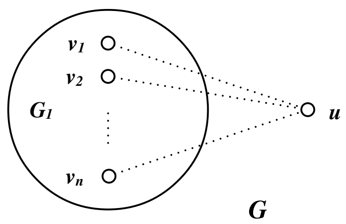

Fig. 1.

Proof: By the Interlacing theorem $\lambda_1(G) > a$ and $\lambda_2(G) \leq a$. Applying the Schwenk's lemma at vertex u, we get the characteristic polynomial of the graph G:

$$P_G(\lambda) = \lambda P_{G_1}(\lambda) - \sum_{i=1}^{m} P_{G_1 - v_i}(\lambda) - 2 \sum_{C \in C(u)} P_{G-C}(\lambda).$$

Now, $\lambda_1(G_1 - v_i) < a$ holds, implying $P_{G_1 - v_i}(a) > 0$ for $i = 1, \dots m$, and also $P_{G-C}(a) > 0$ holds, since graph $G - C$ ($C \in C(u)$) is a subgraph of the graph $G_1 - v_i$ for some $i = 1, \dots m$. Therefore, $P_{G_1}(a) = 0$ implies $P_G(a) < 0$ and $\lambda_2(G) < a$. □

Lemma 3.2 *Let G be a graph with n vertices and the eigenvalues $\lambda_1(G) \geq \lambda_2(G) \geq \lambda_3(G) \geq \dots \geq \lambda_{n-1}(G) \geq \lambda_n(G)$. Let $\alpha = \lambda_{m+1}(G) = \lambda_{m+2}(G) = \dots = \lambda_{m+k}(G)$ be the eigenvalue of multiplicity k. If the polynomial $Q_G(\lambda)$ is defined by the relation $P_G(\lambda) = (\lambda - \alpha)^k Q_G(\lambda)$, then $sgn(Q_G(\alpha)) = (-1)^m$.*

Proof: The characteristic polynomial of the graph G is factorized in the following way: $P_G(\lambda) = \prod_{i=1}^{m}(\lambda - \lambda_i) \cdot (\lambda - \alpha)^k \prod_{i=m+k+1}^{n}(\lambda - \lambda_i)$. Then,

$$Q_G(\lambda) = \prod_{i=1}^{m}(\lambda - \lambda_i) \cdot \prod_{i=m+k+1}^{n}(\lambda - \lambda_i).$$

For $i \in \{1, 2, \dots, m\}$ $\alpha - \lambda_i < 0$, and $sgn\left(\prod_{i=1}^{m}(\alpha - \lambda_i)\right) = (-1)^m$.
For $i \in \{m+k+1, m+k+2, \dots, n\}$ $\alpha - \lambda_i > 0$, and, consequently, $\prod_{i=m+k+1}^{n}(\alpha - \lambda_i) > 0$.
Thus we have proved $sgn(Q_G(\alpha)) = (-1)^m$. □

Here are some simple and useful consequences of the Lemma 3.2. In a connected graph G, if $\lambda_2 = \alpha$, then $Q_G(\alpha) < 0$; or, if $\lambda_3 = \alpha$ and $\lambda_2 > \lambda_3$, then $Q_G(\alpha) > 0$.

Theorem 3.3 *(GT) Let G be the graph in Fig. 2, with a cut-vertex u. Let components of the graph $G - u$, the graphs $G_1, \dots, G_n$, be connected graphs for which $\lambda_2(G_i) \leq a$, $i = 1, \dots, n$, holds. For $a > 0$ it holds:*

1. *If at most one of the graphs $G_1, \dots, G_n$ has index a, and for the rest of them the indices are less than a, then $\lambda_2(G) < a$.*
2. *If at least two of the graphs $G_1, \dots, G_n$ have indices a, and for the rest of them the indices are not greater than a, then $\lambda_2(G) = a$.*
3. *If only one of the graphs $G_1, \dots, G_n$ has index greater than a, and at least one of the remaining graphs has index a, then $\lambda_2(G) > a$.*

Proof: 1. If $\lambda_1(G_1), \lambda_1(G_2), \dots \lambda_1(G_n) < a$, then $\lambda_1\left(\bigcup_{i=1}^{n} G_i\right) < a$ and, therefore, by the Interlacing theorem, $\lambda_2(G) < a$. Now consider the case when exactly one of the graphs G_i has index a, say $\lambda_1(G_1) = a$, and $\lambda_1(G_2), \lambda_1(G_3), \dots \lambda_1(G_n) < a$. $\lambda_1(G_1) = a$ implies $\lambda_2(G_1) < a$. Then, $\lambda_1\left(\bigcup_{i=1}^{n} G_i\right) = a$ and $\lambda_2\left(\bigcup_{i=1}^{n} G_i\right) < a$, and, therefore, by the Interlacing theorem, $\lambda_2(G) \leq a$. $\lambda_1(G_1) = a$ implies $P_{G_1}(a) = 0$, and $\lambda_1(G_2), \lambda_1(G_3), \dots \lambda_1(G_n) < a$

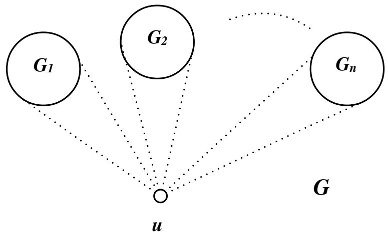

Fig. 2.

implies $P_{G_2}(a), P_{G_3}(a), \ldots P_{G_n}(a) < a$. Applying the Schwenk's lemma at the vertex u of the graph G, we compute the characteristic polynomial of G.

$$\begin{aligned} P_G(a) &= aP_{G_1}(a)\cdot\ldots\cdot P_{G_n}(a) \\ &- \left(\textstyle\sum_{v\in Adju\cap G_1} P_{G_1-v}(a) + 2\sum_{C\in C(u)\cap G_1} P_{G_1-V(C)}(a)\right) P_{G_2}(a)\cdot\ldots\cdot P_{G_n}(a) \\ &- P_{G_1}(a)\left(\textstyle\sum_{v\in Adju\cap(G-G_1)} P_{G-G_1-u-v}(a) + 2\sum_{C\in C(u)\cap(G-G_1)} P_{G-G_1-V(C)}(a)\right) \\ &= -\left(\textstyle\sum_{v\in Adju\cap G_1} P_{G_1-v}(a) + 2\sum_{C\in C(u)\cap G_1} P_{G_1-V(C)}(a)\right) P_{G_2}(a)\cdot\ldots\cdot P_{G_n}(a). \end{aligned}$$

Let us introduce the graph $K = G-(G_2\cup\ldots\cup G_n)$. Then,

$$\begin{aligned} P_K(a) &= aP_{G_1}(a) - \textstyle\sum_{v\in Adju\cap G_1} P_{G_1-v}(a) - 2\sum_{C\in C(u)\cap G_1} P_{G_1-V(C)}(a) \\ &= -\textstyle\sum_{v\in Adju\cap G_1} P_{G_1-v}(a) - 2\sum_{C\in C(u)\cap G_1} P_{G_1-V(C)}(a). \end{aligned}$$

Then, $P_G(a) = P_K(a)\cdot P_{G_2}(a)\cdot\ldots\cdot P_{G_n}(a)$. $P_K(a) < 0$ by the Lemma 3.1, and therefore, $P_G(a) < 0$, implying $\lambda_2(G) < a$.

2. A clear consequence of the Interlacing Theorem.

3. If the index of one of the graphs $G_1,\ldots,G_n$ is greater than a, say $\lambda_1(G_1) > a$, and one is equal to a, say $\lambda_1(G_2) = a$, the Interlacing theorem implies $\lambda_2(G)\geq a$. To prove that a strong inequality $\lambda_2(G) > a$ holds, let us notice that it is sufficient to prove it in the case when graph $G-u$ consists only of the two mentioned components G_1 and G_2. If $\lambda_2(G_1) > a$, then $\lambda_2(G_1\cup G_2) > a$, and, therefore, $\lambda_2(G) > a$. Now, let us consider the case $\lambda_2(G_1) = a$. Let us introduce the graphs $H = G-G_1$ and $K = G-G_2$. Applying the Schwenk's lemma at the vertex u of the graphs H and K we get:

$$\begin{aligned} P_K(\lambda) &= \lambda P_{G_1}(\lambda) - \textstyle\sum_{v\in Adju\cap G_1} P_{G_1-v}(\lambda) - 2\sum_{C\in C(u)\cap K} P_{K-V(C)}(\lambda) \\ P_H(\lambda) &= \lambda P_{G_2}(\lambda) - \textstyle\sum_{v\in Adju\cap G_2} P_{G_2-v}(\lambda) - 2\sum_{C\in C(u)\cap H} P_{H-V(C)}(\lambda). \end{aligned}$$

For the graph G we get:

$$\begin{aligned} P_G(\lambda) &= \lambda P_{G_1}(\lambda)P_{G_2}(\lambda) \\ &- \left(\textstyle\sum_{v\in Adju\cap G_1} P_{G_1-v}(\lambda) + 2\sum_{C\in C(u)\cap K} P_{K-V(C)}(\lambda)\right) P_{G_2}(\lambda) \\ &- \left(\textstyle\sum_{v\in Adju\cap G_2} P_{G_2-v}(\lambda) + 2\sum_{C\in C(u)\cap K} P_{H-V(C)}(\lambda)\right) P_{G_1}(\lambda). \end{aligned}$$

Finally, $P_G(\lambda) = P_K(\lambda)P_{G_2}(\lambda) + P_{G_1}(\lambda)P_H(\lambda) - \lambda P_{G_1}(\lambda)P_{G_2}(\lambda)$. Since $P_{G_2}(a) = 0$, we have $P_G(a) = P_{G_1}(a)P_H(a)$. Since $\lambda_1(G_2) = a$, Lemma 3.1 implies $P_H(a) < 0$.

a) If $\lambda_2(G_1) < a < \lambda_1(G_1)$, then $P_{G_1}(a) < 0$, implying $P_G(a) > 0$, and, therefore, $\lambda_2(G) > a$.

b) Let us consider the case $\lambda_2(G_1) = \lambda_3(G_1) = \ldots = \lambda_k(G_1) = a$ and $\lambda_{k+1}(G_1) < a$, $k \geq 2$. By the Interlacing theorem we have $\lambda_2(K) \geq a$. Similarly, if $\lambda_2(K) > a$, then $\lambda_2(G) > a$.

Now, let us consider the case $\lambda_2(K) = a$. By the Interlacing theorem $\lambda_3(K) = \lambda_4(K) = \ldots = \lambda_k(K) = a$ holds. Let us introduce the polynomial $Q_K(\lambda)$ by $P_K(\lambda) = (\lambda - 1)^{k-1} \cdot Q_K(\lambda)$. Similarly, we have $P_{G_1}(\lambda) = (\lambda - 1)^{k-1} \cdot Q_{G_1}(\lambda)$ and $P_{G_2}(\lambda) = (\lambda - 1) \cdot Q_{G_2}(\lambda)$. Notice that, by Lemma 3.2, $Q_{G_1}(a) < 0$. Now,

$$
\begin{aligned}
P_G(\lambda) &= (\lambda - 1)^{k-1} \cdot Q_K(\lambda) \cdot (\lambda - 1) \cdot Q_{G_2}(\lambda) \\
&+ (\lambda - 1)^{k-1} \cdot Q_{G_1}(\lambda) \cdot P_H(\lambda) \\
&- \lambda \cdot (\lambda - 1)^{k-1} \cdot Q_{G_1}(\lambda) \cdot (\lambda - 1) \cdot Q_{G_2}(\lambda).
\end{aligned}
$$

Let us introduce the polynomial $Q_G(\lambda)$ by $P_G(\lambda) = (\lambda - 1)^{k-1} \cdot Q_G(\lambda)$. Then, $Q_G(a) = Q_{G_1}(a) \cdot P_H(a)$. Since, $Q_{G_1}(a) < 0$ and $P_H(a) < 0$, we get $Q_G(a) > 0$, implying $\lambda_2(G) > a$. □

The Generalized RS-theorem brings improvement of the Interlacing theorem for the cases 1. and 3.

4 Trees whose second largest eigenvalue does not exceed $\frac{\sqrt{5}+1}{2}$

Let us determine all trees T with the property $\lambda_2(T) \leq \frac{\sqrt{5}+1}{2}$, by describing all maximal trees for this property.

The bound $\frac{\sqrt{5}+1}{2}$ has not been considered before. We shall denote it by φ. This number is the greater root of the polynomial $\varphi^2 - \varphi - 1$, hence, it is an index of the path P_4. For every tree one and only one of the following statements holds: 1) a tree is the path P_4, 2) a tree is a proper subgraph of the path P_4, 3) a tree is a proper supergraph of the path P_4, 4) a tree is the star $K_{1,3}$. The only trees for which $\lambda_1 < \varphi$ holds are paths P_1, P_2 and P_3. On the other hand, minimal forbidden trees for the property $\lambda_1 \leq \varphi$ are path P_5 and star $K_{1,3}$ (for both of these trees $\lambda_1 = \sqrt{3}$ holds).

By T_∞ we shall denote the family of trees T with a cut-point u, for which all components of $T - u$ are paths P_1, P_2, P_3 and P_4. For these graphs, by GT, $\lambda_2 \leq \varphi$ holds. Also, we shall say that a tree is GT-decidable for $\lambda_2 = \varphi$ if we can find whether its second largest eigenvalue is less than φ, equal to φ, or greater than φ, only by applying GT to one of its cut-points. All graphs of the family T_∞ are GT-decidable for $\lambda_2 = \varphi$. Graphs whose at least one component is path P_4 are GT-decidable, for $\lambda_2 = \varphi$, too.

In certain cases, the sign of the $P_T(\varphi)$ will be used to compare values $\lambda_2(T)$ and φ. This is explained by the next Lemma, which is a simple consequence of the Interlacing theorem.

Lemma 4.1 *For a tree T, let $\lambda_2(T) < \varphi < \lambda_1(T)$. Let τ be the tree T extended with a pendant edge. Then the following statements hold:*

1. *If $P_\tau(\varphi) < 0$, then $\lambda_2(\tau) < \varphi$.*
2. *If $P_\tau(\varphi) = 0$, then $\lambda_2(\tau) = \varphi$.*
3. *If $P_\tau(\varphi) > 0$, then $\lambda_2(\tau) > \varphi$.*

In the next Lemma we also give values of the $P_T(\varphi)$ for some simple trees.

Lemma 4.2 *1) $P_{P_1}(\varphi) = P_{P_2}(\varphi) = \varphi$, 2) $P_{P_3}(\varphi) = 1$, 3) $P_{P_4}(\varphi) = 0$, 4) $P_{P_{n+5}}(\varphi) = -P_{P_n}(\varphi)$, $n \in \mathbb{N}$, 5) $P_{P_{n+10k}}(\varphi) = P_{P_n}(\varphi)$, $n,k \in \mathbb{N}$, 6) $P_{K_{1,r}}(\varphi) = \varphi^{r-1}(\varphi^2 - r) = \varphi^{r-1}(\varphi + 1 - r)$.*

Proof: These are simple consequences of $P_{P_3}(\varphi) = \prod_{k=1}^{n}(\varphi - 2cos\frac{k\pi}{n+1})$, Schwenk's lemma and the fact that $\varphi^2 = \varphi + 1$. □

Now, we shall describe all trees T with the property $\lambda_2(T) \leq \frac{\sqrt{5}+1}{2}$ that are not GT-decidable for φ.

Theorem 4.3 *Let T be a tree which is not GT-decidable for φ. Then, $\lambda_2(T) \leq \varphi$ if and only if T is a subgraph of some of the trees T_1–T_{59} (Figure 3).*

Proof: Using GT we easily get that $\lambda_2(P_n) > \varphi$, for $n \geq 10$ (P_n is shown in Figure 3, with appropriate labels). In this case, the middle vertex (or one of two such vertices) is considered as a cut-point. Therefore, a diameter of a tree with the property $\lambda_2 \leq \varphi$ must be less than 9. If the diameter of a tree is 8, the tree is GT-decidable for φ. If it is a path P_9, then $\lambda_2 = \varphi$ (cut-vertex is again the middle vertex and two components are paths P_4). If it is a supergraph of P_9, then $\lambda_2 > \varphi$. Let us notice that a tree is GT-decidable for φ (and belongs to the family T_∞) if the diameter of a tree is 2, too. We shall continue by discussing the length of the diameter.

Let $\text{diam}(T) = 7$. Then T contains a path P_8, which we observe as a basic tree, whose vertices (different from end-vertices) may be additionally loaded with some subtrees. At least one of the vertices v_2, v_3 and v_4 (or v_5, v_6 and v_7) must be additionally loaded, otherwise a tree will be GT-decidable for φ. If there is a pendant edge added to any of the vertices v_2, v_3, v_6 or v_7 the tree is GT-decidable, too. The vertex v_4 is considered as a cut-vertex and we get $\lambda_2(T) > \varphi$ (components are $K_{1,3}$ and P_3) or $\lambda_2(T) = \varphi$ (components are two paths P_3). Therefore, there must be at least a pendant edge added to each of the vertices v_4 and v_5. For such tree T $\lambda_2(T) < \frac{\sqrt{5}+1}{2}$ holds. By making further extensions, we can get three different maximal trees, T_1, T_2 and T_3 (Figure 4). For all of them, $\lambda_2 = \varphi$ holds. This can

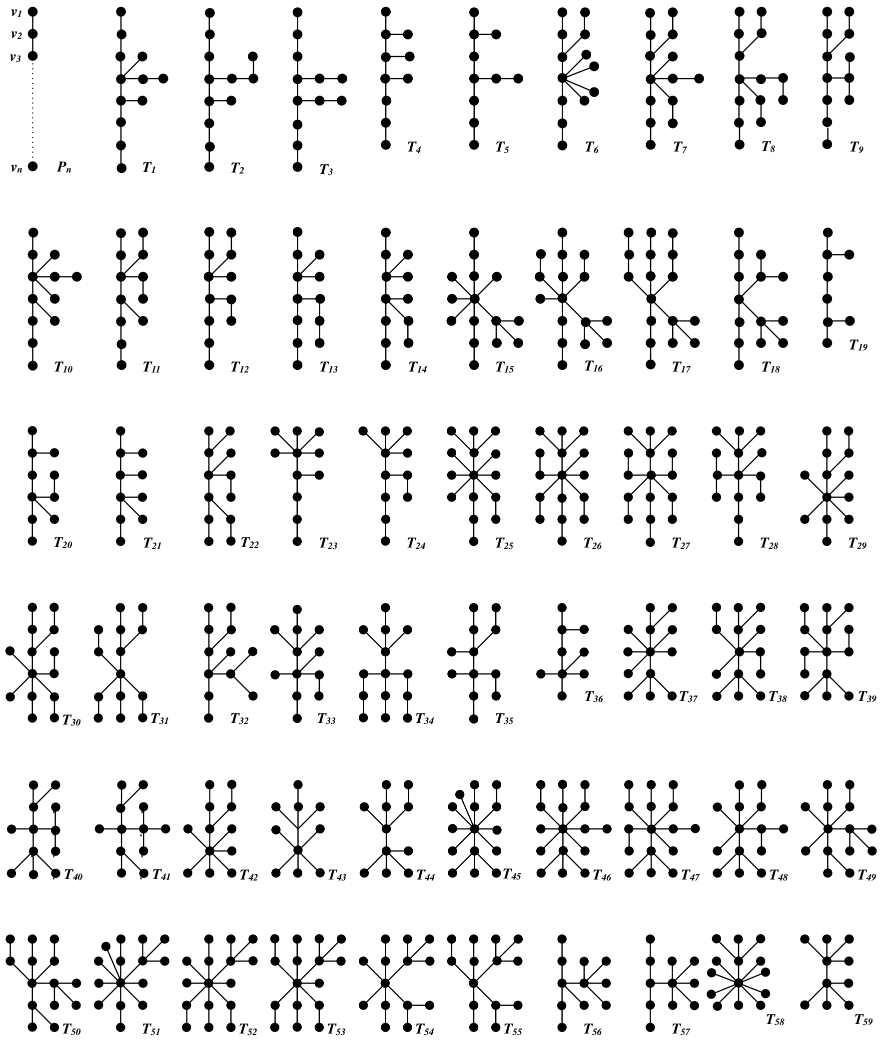

Fig. 3.

be easily proved by using Schwenk's lemma (and also checked by using NEWGRAPH). For example, for the tree T_1, by applying Schwenk's lemma to the vertex v_4 and by using $\varphi^2-\varphi-1=0$, we get $P_{T_1}(\varphi)=-\varphi^3+\varphi^3+\varphi^2+\varphi-\varphi^3=0$. For any subtree of the tree T_1, we can check in the same way that its characteristic polynomial at the point φ is less than zero. Hence, if we add a pendant edge to any vertex of the tree T_1, the characteristic polynomial of this new tree at the point φ shall exceed 0. Therefore, T_1 is a maximal tree for the property $\lambda_2 \leq \frac{\sqrt{5}+1}{2}$.

Let $\mathrm{diam}(T)=6$. The basic tree is the path P_7. At least one of the vertices v_2, v_3 and v_4 (or v_4, v_5 and v_6) must be additionally loaded, otherwise the tree will be GT-decidable for φ. If there is a pendant edge added to a vertex v_2, then the vertices v_5 and v_6 may not be additionally loaded (otherwise, the tree is GT-decidable for φ and $\lambda_2 > \varphi$). Then the vertex v_4 must be additionally loaded. By further extensions and using Lemma 4.1 we can get two maximal trees, T_4 and T_5, for which $\lambda_2 = \varphi$ holds. Let the vertex v_2 be of the degree 2 and let the vertex v_3 be additionally loaded. Then the degrees of the vertices v_5 and v_6 have to be 2, too. Otherwise, if we observe v_4 as a cut-vertex, we get GT-decidable graph. Therefore, the vertex v_4 must be additionally loaded. Further on, we can discuss possible degrees of the vertices v_3 and v_4.

Let $d(v_3) = 3$ and $d(v_4) = r+2$. If there is a pendant edge added to the vertex v_3, the tree becomes GT-decidable for φ. If there is a path P_n ($n > 3$) added to a vertex v_3, the diameter of the graph becomes grater than 6. Therefore, there must be the path P_3 added to the vertex v_3. Let nothing but pendant edges be added to v_4. Then we get $P(\varphi) = \varphi^{r+1} - \varphi^{r+2} + r\varphi^{r-1} + \varphi^{r+1}$. Using $\varphi^2 - \varphi - 1 = 0$, $P(\varphi) = \varphi^{r-1}(r - 2\varphi - 1)$ holds. Therefore, $r \leq 4$. For $r = 4$, $P(\varphi) < 0$. There is no possible extension τ such that $P_\tau(\varphi) < 0$, so we get the maximal tree T_6. For $r = 3$, $r = 2$ and $r = 1$ the only maximal trees that we get by further extending are the trees T_7, T_8 and T_9, respectively.

Let pendant edges be added to the vertices v_3 and v_4 and let $d(v_4) = 3$ and $d(v_3) = r+2$. Applying Schwenk's lemma to v_3 we get $P(\varphi) = -\varphi\varphi^{r+1} + \varphi^{r+1} + r\varphi^r - \varphi^{r+2}$. Using $\varphi^2 - \varphi - 1 = 0$, $P(\varphi) = \varphi^r(r - \varphi - 2)$ holds. Hence, $r \leq 3$. For $r = 3$, the only possible extension is the tree T_{10}. For $r = 2$, there are four possible extensions: T_{11}, T_{12}, T_{13} and T_{14}.

If the only loaded vertex is v_4, then this vertex must be an end-vertex of a bridge which connects the basic tree P_7 and another tree τ whose index is greater than φ. If the tree τ is the path P_4, then it must be leaned on its middle vertex, otherwise the diameter becomes greater than 6. For the whole tree $\lambda_2 < \varphi$ holds, but the only possible extension leads to the tree T_8. If the tree τ is the star $K_{1,3}$, it must be leaned on its middle vertex (otherwise, $\lambda_2 > \varphi$). Hence, λ_2 of the whole tree is less than φ, and this case allows 4 new extensions: the trees T_{15}–T_{18}.

Let $\mathrm{diam}(T) = 5$. The basic tree is the path P_6. At least one of the vertices v_2, v_3 and v_4 (or v_3, v_4 and v_5) must be additionally loaded, otherwise the tree will be GT-decidable for φ. If there is a pendant edge added to the vertex v_2, then at least one of the vertices v_3, v_4 or v_5 has to be additionally loaded. If it is v_5, we get maximal tree T_{19}. If it is v_4 and if it is loaded with a pendant edge, the tree becomes GT-decidable. Let $d(v_4) = 4$ (for $d(v_4) = 5$, $P(\varphi)$ becomes greater than 0). There are two possible maximal extensions – T_{20} and T_{21}. In the case of $d(v_4) = 3$, there has to be a path P_3 leaned on the vertex v_4 and the only maximal extension is T_{22}. Now let only v_2 and v_3 be loaded. Let $d(v_2) = i$ and $d(v_3) = j$. Using Scwenk's lemma we get $P(\varphi) = \varphi^{2+i}(-j\varphi^{j-1}) - (i+1)\varphi^i(-j\varphi^{j-1}) - \varphi^{i+j+1} = \varphi^{i+j+1}((i+1)j - (j+1)(\varphi+1))$. Hence, if there is only a pendant edge leaned on v_3, maximum degree of v_2 is 6 and the tree T_{23} is maximal. For $d(v_2) = 5$, the only maximal extension is T_{24}. For $d(v_2) = 4$, we get 4 maximal extensions: T_{25}–T_{28}. For $d(v_2) = 3$, v_3 must be an end-vertex of a bridge and the other end-vertex must be the middle vertex of the star $K_{1,3}$ (otherwise, the tree remains GT-decidable, or its diameter exceeds 5). But this case produces only maximal tree T_{28}. Let $d(v_2) = d(v_5) = 2$, $d(v_3) = i+2$ and $d(v_4) = j+2$. As before, we get $P(\varphi) = \varphi^{i+j}(ij - i - j - \varphi)$. Hence, $min(i,j) \leq 2$. Let $i \leq j$. If $i = 1$ there has to be the path P_3 leaned on v_3 and then $j \leq 5$ holds. In this case, there are 4 maximal trees T_{29}–T_{32}. For $i = 2$, there are 3 maximal trees T_{33}–T_{35}.

If v_3 is the only loaded vertex, then it must be an end-vertex of a bridge which connects the basic tree P_6 and another tree τ whose index is greater than φ. Because of $\mathrm{diam}(T) = 5$, the tree τ is the star $K_{1,3}$. For the whole tree $P(\varphi) < 0$ holds, but further extending brings only trees that are already described - T_{23}–T_{28}.

Let $\mathrm{diam}(T) = 4$. The basic tree is the path P_5. Vertices v_2 and v_4 are to be loaded only by pendant edges. Let $d(v_2) = i$, $d(v_3) = 2$ and $d(v_4) = j$. As before, using Schwenk's lemma we get $P(\varphi) = \varphi^{i+j-3}(ij - (i+j)(\varphi+1) + 3\varphi + 1)$, and hence, $min(i,j) \leq 3$. Otherwise the tree becomes GT-decidable for φ. Let $i \leq j$. If $i = 3$, then $j \leq 5$. Let $i = 3$: for $j = 5$, we get maximal tree T_{36}; for $j = 4$, the vertex v_3 can be additionally loaded and by further extending we get maximal trees T_{37}–T_{40}. For $j = 3$, there must be a bridge whose end-vertex is v_3 and the star $K_{1,3}$ is leaned on the other end. This tree is not maximal, but it can be extended in only one way, which gives maximal tree T_{41}. If $j = 2$, then $d(v_3) \geq 3$. For $d(v_3) = 3$, the path P_3 must be leaned on the vertex v_3 and $P(\varphi) = \varphi^{j-2}(j - 3\varphi - 3)$ holds, and hence, $j \leq 7$. For $j = 7$ we get maximal tree T_{42}. Let $d(v_3) = k \geq 4$. Then from $P(\varphi) = \varphi^{j+k-4}(kj - 3j - 2k + 5 + 2\varphi - k\varphi)$ we get $j \leq 6$. For $j = 6$ there is only one maximal tree T_{43}, for $j = 5$ only one maximal tree T_{44}, while for $j = 4$ we get 4 maximal trees T_{45}–T_{48}. For $j = 3$ and $j = 2$, there must be a bridge to the star $K_{1,3}$ leaned on the vertex v_3 which leads to maximal trees T_{49}–T_{57}.

Let $\mathrm{diam}(T) = 3$. Then there are only pendant edges leaned on the vertices v_2 and v_3. Let $d(v_2) = i - 2$ and $d(v_3) = j - 2$, hence $min(i,j) \geq 4$ (otherwise such a tree belongs to the family T_∞). As before, we get $P(\varphi) = \varphi^{i+j-4}((i-2)(j-2) - \varphi(i+j-4))$. Therefore, $min(i,j) \leq 5$. Let $i \leq j$: for $i = 4$, we get $j_{max} = 10$ and for $i = 5$, $j_{max} = 5$, which brings maximal trees T_{58} and T_{59}. □

From the previous results it follows the main theorem.

Theorem 4.4 *A tree T has the property $\lambda_2(T) \leq \frac{\sqrt{5}+1}{2}$ if and only if it belongs to the family of trees T_∞ or it is a subgraph of some of the trees T_1–T_{59}.*

Acknowledgement

The programming package newGRAPH [14] was used for computations of eigenvalues of graphs.

References

[1] D. CAO and Y. HONG, *Graphs characterized by the second eigenvalue*, J. Graph Theory Vol. 17, 3 (1993), 325–331.

[2] D. M. CVETKOVIĆ, M. DOOB, H. SACHS, *Spectra of Graphs-Theory and Application*, Deutscher Verlag der Wissenschaften-Academic Press, Berlin-New York, 1980; second edition 1982; third edition, Johann Ambrosius Barth Verlag, Heidelberg-Leipzig, 1995.

[3] D. CVETKOVIĆ and S. SIMIĆ, *On graphs whose second largest eigenvalue does not exceed* $\frac{\sqrt{5}-1}{2}$, Discrete Math. Vol. 138 (1995), 213–227.

[4] T. KOLEDIN and Z. STANIĆ, *Regular graphs whose second largest eigenvalue is at most* 1, Novi Sad J. Math., Vol. 43, 3 (2013), 145–153.

[5] T. KOLEDIN and Z. STANIĆ, *Regular graphs with small second largest eigenvalue*, Appl. Anal. Discrete Math. Vol. 7 (2013), 235–249.

[6] M. PETROVIĆ and Z. RADOSAVLJEVIĆ, *Spectrally constrained graphs*, Fac. of Science, Kragujevac, Serbia, 2001.

[7] Z. RADOSAVLJEVIĆ, B. MIHAILOVIĆ and M. RAŠAJSKI, *Decomposition of Smith graphs in maximal reflexive cacti*, Discrete Math., Vol. 308, 2-3 (2008), 355–366.

[8] Z. RADOSAVLJEVIĆ, B. MIHAILOVIĆ and M. RAŠAJSKI, *On bicyclic reflexive graphs*, Discrete Math., Vol. 308, 5-6 (2008), 715–725.

[9] Z. RADOSAVLJEVIĆ and M. RAŠAJSKI, *Multicyclic treelike reflexive graphs*, Discrete Math., Vol. 296, 1 (2005), 43–57.

[10] Z. RADOSAVLJEVIĆ and S. SIMIĆ, *Which bicyclic graphs are reflexive?*, Univ. Beograd, Publ. Elektrotehn. Fak., Ser. Mat. Vol. 7 (1996), 90–104.

[11] M. RAŠAJSKI, *On a class of maximal reflexive* θ*-graphs generated by Smith graphs*, Appl. Anal. Discrete Math. Vol. 1 (2007), 241–250.

[12] M. RAŠAJSKI, Z. RADOSAVLJEVIĆ and B. MIHAILOVIĆ , *Maximal reflexive cacti with four cycles: The approach via Smith graphs*, Linear Algebra Appl. Vol. 435 (2011), 2530–2543.

[13] J. H. SMITH, *Some properties of the spectrum of a graph* In: Combinatorial Structures and Their Applications (R. Guy, H. Hanani, N. Sauer, J. Schonheim) Gordon and Breach, Science Publ., Inc., New York-London-Paris 1970, pp. 403–406.

[14] D. STEVANOVIĆ, V. BRANKOV, D. CVETKOVIĆ and S. Simić, newGRAPH, the expert system. http://www.mi.sanu.ac.rs/newgraph.