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Merrifield–Simmons and Hosoya Index of Thorn–regular Graphs

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Abstract: This paper deals with the Merrifield–Simmons and Hosoya indices of some thorn graphs. We outline a method for the calculation of these indices in the case of regular caterpillars and regular cyclic caterpillars. Then, by using a result of Belbachir & Bencherif, [*Linear recurrent sequences and powers of a square matrix*, Integers, Vol. 6, 2006, #A12], we obtain combinatorial expressions for these indices.

Keywords: Merrifield–Simmons index, Hosoya index, thorn graph, independent vertex sets, independent edge sets

1 Introduction

Let $G = (V,E)$ be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Two vertices of G are said to be independent if they are not adjacent in G. Two edges of G are said to be independent if they have no common edge-vertex (i.e., if they are no incident). For a vertex $v \in V(G)$, denote by $N(v)$ the set of vertices of the graph G consisting of all first neighbors of v. Let further, $N[v] = N(v) \cup \{v\}$.

A subset of $V(G)$ is said to be an independent vertex set if no two vertices in the subset are adjacent. The Merrifield–Simmons index of the graph G, denoted by $\sigma(G)$, is defined as the total number of independent vertex sets of G, including the empty set.

A subset of $E(G)$ is said to be an independent edge set if no two edges in the subset are incident. The Hosoya index of the graph G, denoted by $Z(G)$, is defined as the total number of independent edge sets of G. The Hosoya index is equal to the total number of matchings in G, including the empty set.

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Denote by F_n is the n-th Fibonacci number and recall that $F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$ with initial conditions $F_0 = 1$ and $F_1 = 1$, and by L_n is the n-th Lucas number $L_n = L_{n-1} + L_{n-2}$ $(n \geq 2)$ with initial conditions $L_0 = 2$ and $L_1 = 1$.

The Merrifield–Simmons and Hosoya indices are well studied in the literature and were computed for many classes of graphs. Details on their theory as well as additional references can be found in the survey [16]. In particular, see [10] and [19] for trees, [17] for trees with given number of pendent vertices, [9] for trees with a given diameter, [4] for trees with bounded degree, [12] for unicyclic graphs, [8] for the unicyclic graphs with a given diameter, and [6] for unicyclic graphs with given number of cut vertices.

Among all n-vertex trees, the star S_n has the greatest Merrifield–Simmons index and the path P_n has the smallest Merrifield–Simmons index, such that for any n-vertex tree T, $F_{n+1} = \sigma(P_n) \leq \sigma(T) \leq \sigma(S_n) = 2^{n-1} + 1$ [16]. The analogous result for the Hosoya index reads [16]: $n = Z(S_n) \leq Z(T) \leq Z(P_n) = F_{n+1}$.

The concept of thorny graphs was introduced by one of the present authors in 1998 [3], and has eventually been used in the literature for a variety of applications [2, 5, 7, 13, 14, 15].

In this paper, we examine the Merrifield–Simmons and Hosoya indices for some thorny graphs, namely of regular caterpillars and regular cyclic caterpillars. Moreover, some combinatorial interpretation are provided.

The paper is organized as follows. Section 2 presents some basic definitions related to graphs. In section 3, we give two characterizations of the Merrifield–Simmons index of regular caterpillars. In section 4, we compute the Merrifield–Simmons index of cyclic caterpillars. Finally, in section 5, we are interested in the calculation of the Hosoya index of regular caterpillars and cyclic caterpillars.

2 Definitions

Definition 2.1 *Let T be a tree of order n. Then, the thorny tree T^* is obtained from T by attaching p_i new vertices to each vertex v_i of T, $i = 1, \ldots, n$. Thus, the number of vertices of T^* is $n^* = n + \sum_{i=1}^{n} p_i$. If $p_i = p$ for all $i = 1, \ldots, n$ then the thorny tree is said to be thorn-regular.*

Definition 2.2 *An acyclic graph T^* is called a caterpillar if the deletion of all its pendent vertices reduces it to a path.*

Definition 2.3 *A unicyclic graph C^* is called cyclic caterpillar if the deletion of all its pendent vertices reduces it to a cycle.*

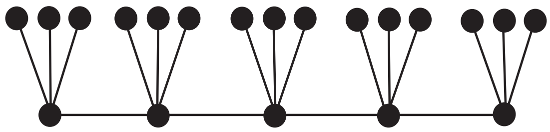

Fig. 1. A thorn-regular caterpillar.

3 Merrifield–Simmons index of thorn-regular caterpillars

3.1 A first characterization

Let T_n^* be a thorn-regular caterpillar with p vertices attached to each vertex v_i, $i = 1,\ldots,n$ of the parent tree T.

Using Lemma 3.1 below, we will obtain a first characterization of the Merrifield–Simmons index of thorn-regular caterpillars.

Lemma 3.1 [11, 16] *For any graph G and any of its vertices v,*

$$\sigma(G) = \sigma(G-v) + \sigma(G-N[v])\,.$$

Theorem 3.2 *Let T^* be a thorn-regular caterpillar. Then*

$$\sigma(T_n^*) = 2^p\,\sigma(T_{n-1}^*) + 2^p\,\sigma(T_{n-2}^*)\ \ ,\ \ n \geq 2 \tag{1}$$

with initial conditions $\sigma(T_0^*) = 1$ and $\sigma(T_1^*) = 2^p + 1$.

Proof Apply Lemma 3.1, and choose in T_n^* a vertex of degree $p+1$. Then

$$T_n^* - v = T_{n-1}^* \cup E_p \qquad \text{and} \qquad T_n^* - N[v] = T_{n-2}^* \cup E_p$$

where E_p stands for an edgeless graph with p vertices. Theorem 3.2 follows now from $\sigma(G_1 \cup G_2) = \sigma(G_1) \cdot \sigma(G_2)$ and $\sigma(E_p) = 2^p$.

□

Corollary 3.3 *For $p = 0$ it holds $\sigma(T_n^*) = \sigma(P_n) = F_n$ with $\sigma(T_0^*) = 1$ and $\sigma(T_1^*) = 2$.*

Theorem 3.4 *Let T^* be a thorn-regular caterpillar. Then*

$$\sigma(T_n^*) = \left(\frac{1}{2} - \frac{2^p+2}{\sqrt{2^p\,(2^p+4)}}\right)\left(\frac{2^p - \sqrt{2^p\,(2^p+4)}}{2}\right)^n + \left(\frac{1}{2} - \frac{2^p+2}{\sqrt{2^p\,(2^p+4)}}\right)\left(\frac{2^p + 2\sqrt{2^p\,(2^p+4)}}{2}\right)^n .$$

Proof By Theorem 3.2 we have the recurrence relation

$$\sigma(T_n^*) = 2^p\,\sigma(T_{n-1}^*) + 2^p\,\sigma(T_{n-2}^*) .$$

Its characteristic equation is

$$x^2 - 2^p\,x - 2^p = 0$$

whose zeros are

$$r_1 = \frac{1}{2}\left[2^p - \sqrt{2^p\,(2^p+4)}\right] \qquad \text{and} \qquad r_2 = \frac{1}{2}\left[2^p + \sqrt{2^p\,(2^p+4)}\right] .$$

This means that

$$\sigma(T_n^*) = \alpha\, r_1^n + \beta\, r_2^n .$$

Then we deduce α and β from the initial conditions $\sigma(T_0^*) = 1$ and $\sigma(T_1^*) = 2^p + 1$, resulting in:

$$\alpha = \frac{1}{2} - \frac{2^p + 2}{\sqrt{2^p\,(2^p+4)}} \qquad \text{and} \qquad \beta = \frac{1}{2} + \frac{2^p + 2}{\sqrt{2^p\,(2^p+4)}}.$$

□

Corollary 3.5 *For $p = 0$, we have*

$$\sigma(P_n) = \frac{\sqrt{5}-6}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n + \frac{\sqrt{5}+6}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

3.2 A combinatorial representation

In this section, we give some combinatorial representations of the Merrifield–Simmons index of thorn-regular caterpillars. In the paper by Belbachir and Bencherif [1], the following results were obtained:

Theorem 3.6 [1] *Let $(u_n)_{n>-m}$ be the sequence of elements defined by:*

$$\begin{cases} u_{-j} = 0 & \text{if } 1 \le j \le m-1 \\ u_0 = 1 & \\ u_n = a_1\,u_{n-1} + a_2\,u_{n-2} + \cdots + a_m\,u_{n-m} & \text{if } n \ge 1. \end{cases}$$

Then for all integers $n > -m$,

$$u_n = \sum_{k_1+2k_2+\cdots+mk_m=n} \binom{k_1+k_2+\cdots+k_m}{k_1,k_2,\ldots,k_m} a_1^{k_1} a_2^{k_2} \ldots a_m^{k_m}.$$

Corollary 3.7 [1] *Let $q \ge 1$ be an integer, a, b elements of a unitary commutative ring, and let $(v_n)_{n\ge -q}$ be the sequence of elements defined by:*

$$\begin{cases} v_{-j} = 0 & \text{if } 1 \le j \le q \\ v_0 = 1 & \\ v_n = a\,v_n + b\,v_{n-q} & \text{if } n \ge 0. \end{cases}$$

Then for all integers $n \geq -q$,

$$v_n = \sum_{k=0}^{\lfloor \frac{n}{q+1} \rfloor} \binom{n-kq}{k} a^{n-k(q+1)} b^k$$

whereas for all $n \geq 0$,

$$v_{n+1} + b\,v_{n-q} = 2v_{n+1} - a\,v_n = \sum_{k=0}^{\lfloor \frac{n+1}{q+1} \rfloor} \frac{n+1-k(q-1)}{n+1-kq} \binom{n+1-kq}{k} a^{n+1-k(q+1)} b^k \,.$$

From Theorem 3.6 and Corollary 3.7, we arrive at:

Theorem 3.8 *Let T_n^* be a thorn-regular caterpillar. Then*

$$\sigma(T_n^*) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j}{j} 2^{p(n-j)} \,.$$

Proof Setting $\sigma(T_n^*) = u_n$, we get

$$\begin{cases} u_{-1} & = & 0 \\ u_0 & = & 1 \\ u_n & = & 2^p\,u_{n-1} + 2^p\,u_{n-2} \;\;,\;\; n \geq 1 \end{cases}$$

which implies

$$u_n = \sum_{i+2j=n} \binom{i+j}{i} 2^{p(i+j)} \,.$$

□

3.3 A second characterization

Using the lemma below, we can obtain another characterization of the Merrifield–Simmons index of thorn-regular caterpillars.

Lemma 3.9 [3] *If $uv \in E(G)$, then*

$$\sigma(G) = \sigma(G-uv) - \sigma\Big(G - (N(u) \cup N(v))\Big)\,. \tag{2}$$

Let $uv \in E(T_n^*)$ be an extremal edge. Then we have:

Theorem 3.10 *Let T_n^* be a thorn-regular caterpillar with $uv \in E(T_n^*)$ then*

$$\sigma(T_n^*) = (2^p+1)\,\sigma(T_{n-1}^*) - 2^p\,\sigma(T_{n-3}^*) \;\;,\;\; n \geq 3$$

with $\sigma(T_0^*) = 1$, $\sigma(T_1^*) = 2^p+1$, and $\sigma(T_2^*) = 2^p\,(2^p+2)$.

Proof If $uv \in E(T_n^*)$ is an extremal edge, then by applying the Eq. (2), we obtain that

$$\sigma(T_n^* - uv) = \sigma(S_{p+1})\,\sigma(T_{n-1}^*) = (2^p+1)\,\sigma(T_{n-1}^*)$$

and in addition,

$$\sigma\Big(T_n^* - (N(u) \cup N(v))\Big) = 2^p\,\sigma(T_{n-3}^*)\,.$$

Then

$$\sigma(T_n^*) = (2^p+1)\,\sigma(T_{n-1}^*) - 2^p\,\sigma(T_{n-3}^*) \;\;,\;\; n \geq 3$$

with initial conditions $\sigma(T_0^*) = 1$, $\sigma(T_1^*) = 2^p+1$, and $\sigma(T_2^*) = 2^p\,(2^p+2)$.

□

Corollary 3.11 *For $p = 0$, we have*

$$\sigma(T_n^*) = \sigma(P_n) = 2\,\sigma(P_{n-1}) - \sigma(P_{n-3}) \;\;,\;\; n \geq 3$$

with $\sigma(P_0) = 1$, $\sigma(P_1) = 2$ and $\sigma(P_2) = 3$.

Corollary 3.12 *For $p = 0$, we have*

$$\sigma(P_n) = F_n = 2\,F_{n-1} - F_{n-3} \;\;,\;\; n \geq 3\,.$$

Theorem 3.13 *Let T_n^* be a thorn-regular caterpillar. Then*

$$\sigma(T_n^*) = 1 + \lambda_2\, r_1^n + \lambda_3\, r_2^n$$

for

$$\lambda_2 = \frac{[2^p(2^p+2)-1](1-r_1)\,2^p}{(r_2-1)^2-(r_1-1)(r_2-1))} \quad ; \quad \lambda_3 = \frac{[2^p(2^p+2)-1](1-r_2)\,2^p}{(r_1-1)^2-(r_1-1)(r_2-1))}$$

$$r_1 = \frac{1}{2}\left[2^p - \sqrt{2^p\,(2^p+4)}\right] \quad ; \quad r_2 = \frac{1}{2}\left[2^p + \sqrt{2^p\,(2^p+4)}\right].$$

Proof Use the recurrence relation from Theorem 3.10, whose that The characteristic equation is:

$$x^3 - (2^p+1)x^2 + 2^p = 0\,.$$

Its roots are $\lambda_1 = 1$,

$$\lambda_2 = \frac{[2^p\,(2^p+2)-1][(1-r_1)\,2^p]}{(r_2-1)^2-(r_1-1)(r_2-1))} \quad \text{and} \quad \lambda_3 = \frac{[2^p\,(2^p+2)-1][(1-r_2)\,2^p]}{(r_1-1)^2-(r_1-1)(r_2-1))}\,.$$

Theorem 3.13 follows now from the initial conditions $\sigma(T_0^*) = 1$, $\sigma(T_1^*) = 2^p+1$, and $\sigma(T_3^*) = 2^p(2^p+2)$.

□

Belbachir and Bencherif [1] proved also the following result:

Theorem 3.14 [1] *Let $(u_n)_{n>-m}$ be the sequence defined by:*

$$\begin{cases} u_{-j} &= \alpha_j, \text{ for } 0 \le j \le m-1 \\ u_n &= a_1\, u_{n-1} + a_2\, u_{n-2} + \cdots + a_m\, u_{n-m} \ , \ n \ge 1. \end{cases}$$

Let $(\lambda_j)_{0\le j\le m-1}$, $(y_n)_{n>-m}$ be the sequences of elements of A defined by

$$\lambda_j = -\sum_{k=j}^{m-1} a_{k-j}\,\alpha_k \ , \ 0 \le j \le m-1 \ , \ a_0 = -1$$

$$y_n = \sum_{k_1+2k_2+\cdots+mk_m=n} \binom{k_1+k_2+\cdots+k_m}{k_1,k_2,\ldots,k_m} a_1^{k_1} a_2^{k_2} \ldots a_m^{k_m} \ , \ n > -m\,.$$

Then for all integers $n > -m$,

$$u_n = \lambda_0 y_n + \lambda_1 y_{n+1} + \cdots + \lambda_{m-1} y_{n+m-1} .$$

Theorem 3.15 *Let T_n^* be a thorn-regular caterpillar. Then*

$$\sigma(T_n^*) = 2^{-p} y_n + 2^p y_{n+1} \tag{3}$$

with

$$y_n = \sum_{k=0}^{\lfloor n/3 \rfloor} \binom{n-k}{k} (2^p+1)^{n-3k} 2^{kp} . \tag{4}$$

Proof From Theorem 3.10 we have

$$\begin{cases} \sigma(T_n^*) &= (2^p+1)\,\sigma(T_{n-1}^*) - 2^p\,\sigma(T_{n-3}^*) \ , \ n \geq 3 \\ \sigma(T_0^*) &= 1\,,\, \sigma(T_1^*) = (2^p+1)\,,\, \sigma(T_2^*) = 2^p\,(2^p+2)\,. \end{cases}$$

By Theorem 3.13 with $m = 3$ and $0 \leq j \leq 2$, we obtain the following system:

$$\begin{cases} \sigma(T_n^*) &= a_1\,\sigma(T_{n-1}^*) + a_2\,\sigma(T_{n-2}^*) + a_3\,\sigma(T_{n-3}^*) \\ \sigma(T_0^*) &= 1\,,\, \sigma(T_{-1}^*) = 2^{-p}\,,\, \sigma(T_{-2}^*) = 0\,. \end{cases}$$

Then we compute λ_j and get $\lambda_0 = -2^{-p}$, $\lambda_1 = 2^{-p}$, and $\lambda_2 = 0$.

Hence

$$y_n = \sum_{i+3k=n} \binom{i+k}{i,k} (2^p+1)^i 2^{kp} .$$

With a change of variables, we obtain Eq. (4), resulting in Eq. (3).

□

4 Hosoya index of thorn-regular caterpillars

We known from [16] that

$$Z(G) = Z(G - uv) + Z(G - \{u, v\}) \tag{5}$$

for $uv \in E(G)$, and

$$Z(G) = Z(G - v) + \sum_{w \in N(v)} Z(G - \{w, v\}) \tag{6}$$

for $v \in V(G)$.

In addition, $Z(S_n) = n$ and $Z(K_n) = z_n$, where $z_n = z_{n-1} + (n-1)z_{n-2}$ with $z_1 = 1$ and $z_2 = 2$.

We also know that the Hosoya index of an edgeless graph is equal to one by convention. So, we have the following result:

Lemma 4.1 *Let T_n^* be a thorn-regular caterpillar. Then*

$$Z(T_n^* - \{w, v\}) = \begin{cases} Z(T_{n-1}^*) & \text{if } w \text{ is a thorn vertex} \\ Z(T_{n-2}^*) & \text{else.} \end{cases}$$

Proof Let v be an extremal vertex, i.e, a vertex of a path in the thorn-regular caterpillar, and w a vertex incident to v. So, w is either a thorn vertex or a path vertex. Assume that w is a thorn vertex.

It is clear that the deletion of both v and w induces a graph

$$T_n^* - \{w, v\} = T_{n-1}^* \cup \{v_1, v_2, \dots, v_{p-1}\}$$

with $\{v_1, v_2, \dots, v_{p-1}\}$ an independent set. Hence

$$Z(T_n^* - \{w, v\}) = Z(T_{n-1}^*) \,.$$

Else, w is a vertex of a path, and then we have that deleting both v and w induces a graph

$$T_n^* - \{w, v\} = T_{n-2}^* \cup \{v_1, v_2, \ldots, v_p\} \cup \{w_1, w_2, \ldots, w_p\}$$

with $\{v_1, v_2, \ldots, v_p\}$ and $\{w_1, w_2, \ldots, w_p\}$ being independent sets. Thus

$$Z(T_n^* - \{w, v\}) = Z(T_{n-2}^*) .$$

□

Theorem 4.2 *Let T_n^* be a thorn-regular caterpillar. Then*

$$Z(T_n^*) = (p+1)\, Z(T_{n-1}^*) + Z(T_{n-2}^*) .$$

Proof Use Eqs. (5) and/or (6). Any of these gives the same results. The first one implies that

$$Z(T_n^*) = Z(T_{n-1}^*)\, Z(S_{p+1}) + Z(T_{n-2}^*)$$

and thus

$$Z(T_n^*) = (p+1)\, Z(T_{n-1}^*) + Z(T_{n-2}^*) . \qquad (7)$$

The second relation implies that

$$Z(T_n^*) = Z(T_{n-1}^*) + \sum_{w \in N(v)} Z(T_n^* - \{v, w\})$$

and using Lemma 4.1, we again arrive at (7).

□

Corollary 4.3 *For $p = 0$, we have*

$$Z(T_n^*) = Z(P_n) = F_{n-1}$$

with $Z(T_0^*) = 1$ and $Z(T_1^*) = 2$.

In an analogous manner as Theorems 3.4 and 3.13, we prove:

Theorem 4.4 *Let T_n^* be a thorn-regular caterpillar. Then*

$$Z(T_n^*) = \alpha \left(\frac{p+1-\sqrt{(p+1)^2+4}}{2} \right)^n + \beta \left(\frac{p+1+\sqrt{(p+1)^2+4}}{2} \right)^n$$

with

$$\alpha = \frac{1}{2} - \frac{p+1}{2\sqrt{p+1)^2+4}} \qquad \text{and} \qquad \beta = \frac{1}{2} + \frac{p+1}{2\sqrt{p+1)^2+4}}\,.$$

Corollary 4.5

$$Z(P_n) = \frac{\sqrt{5}-1}{2\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n + \frac{\sqrt{5}+1}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n .$$

5 Merrifield–Simmons index of thorn-regular cyclic caterpillars

Let C_n^* be a thorn-regular cyclic caterpillar with p vertices attached to each vertex v_i, $i = 1,\dots,n$.

Using the lemma below, we will obtain a characterization for the Merrifield–Simmons index of thorn-regular cyclic caterpillars.

Lemma 5.1 [18] *If $uv \in E(G)$, then $\sigma(G) = \sigma(G-uv) - \sigma\Big(G - N(u) \cup N(v)\Big)$.*

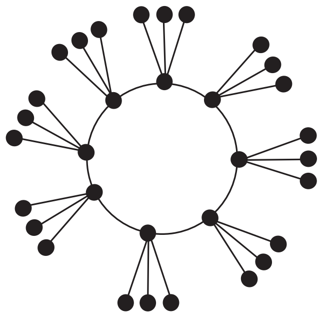

Fig. 2. A thorn-regular cyclic caterpillar.

Theorem 5.2 *Let C_n^* be a thorn-regular cyclic caterpillar. Then*

$$\sigma(C_n^*) = \sigma(T_n^*) - 2^{2p}\,\sigma(T_{n-4}^*) \quad , \quad n \geq 3$$

where $\sigma(C_0^*) = 1$ and $\sigma(C_1^*) = 2^p + 1$.

Proof We have from Lemma 5.1 that for $uv \in E(C_n^*)$,

$$\sigma(C_n^*) = \sigma(C_n^* - uv) - \sigma\Big(C_n^* - N(u) \cup N(v)\Big)\,.$$

Then

$$\sigma(C_n^*) = \sigma(T_n^*) + 2^p\,\sigma(T_{n-4}^*)\,. \tag{8}$$

□

Corollary 5.3 *For $p = 0$,*

$$\sigma(C_n^*) = L_n \quad , \quad n \geq 3$$

with $\sigma(C_0^*) = L_0 = 1$ and $\sigma(C_1^*) = L_1 = 2$.

Theorem 5.4 *Let C_n^* be a thorn-regular cyclic caterpillar. Then*

$$\sigma(C_n^*) = 2^p\,\sigma(C_{n-1}^*) + 2^p\,\sigma(C_{n-2}^*)$$

where $\sigma(C_0^*) = 1$ and $\sigma(C_1^*) = 2^p + 1$.

Proof Combine Eqs. (1) and (8).

□

Corollary 5.5 *Let C_n^* be a thorn-regular cyclic caterpillar then*

$$\sigma(C_n^*) = \alpha\left(\frac{2^p - \sqrt{2^p(2^p+4)}}{2}\right)^n + \beta\left(\frac{2^p + 2\sqrt{2^p(2^p+4)}}{2}\right)^n$$

with

$$\alpha = \frac{1}{2} - \frac{2^p+2}{\sqrt{2^p(2^p+4)}} \qquad \textit{and} \qquad \beta = \frac{1}{2} + \frac{2^p+2}{\sqrt{2^p(2^p+4)}}\,.$$

Corollary 5.6 *Let C_n^* be a thorn-regular cyclic caterpillar. Then*

$$\sigma(C_n^*) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j}{j} 2^{p(n-j)}\,.$$

Theorem 5.7 *Let C_n^* be a thorn-regular cyclic caterpillar. Then*

$$\sigma(C_n^*) = \sum_{j=0}^{\lfloor n/2 \rfloor} \left[1 + \frac{1}{2^p}\frac{j(j-1)}{(n-j)(n-j-1)}\right] \binom{n-j}{j} 2^{p(n-j)}\,.$$

Proof Let $\sigma(C_n^*) = u_n$. Then

$$\begin{cases} u_{-1} &= 0 \\ u_0 &= 1 \\ u_n &= 2^p u_{n-1} + 2^p u_{n-2} \ , \ n \geq 1 \, . \end{cases}$$

Then by applying Corollary 3.7, we obtain:

$$u_n = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j}{j} 2^{p(n-j)} + \sum_{k=0}^{\lfloor (n-4)/2 \rfloor} \binom{n-4-k}{k} 2^{p(n-4-k)}$$

Then by the change of variables $j = k+2$, we obtain a result.

□

6 Hosoya index of thorn-regular cyclic caterpillars

Let C_n^* be a thorn-regular cyclic caterpillar with p vertices attached to each vertex v_i, $i = 1,\ldots,n$. Using $Z(G) = Z(G-uv) + Z(G-\{u,v\})$, we have:

Theorem 6.1 *Let C_n^* be a thorn-regular cyclic caterpillar, then we have*

$$Z(C_n^*) = (p+1)Z(C_{n-1}^*) + Z(C_{n-2}^*) \, .$$

Proof We have that $Z(G) = Z(G-uv) + Z(G-\{u,v\})$ and therefore

$$Z(C_n^*) = Z(T_n^*) + Z(T_{n-2}^*) \, .$$

Application of Theorem 4.2 by replacing $Z(T_n^*)$ and $Z(T_{n-2}^*)$ yields a result.

□

Since $Z(T_n^*)$ and $Z(C_n^*)$ have the same characteristic polynomial, we have

Corollary 6.2

$$Z(C_n^*) = \alpha \left(\frac{(p+1) - \sqrt{(p+1)^2+4}}{2} \right)^n + \beta \left(\frac{(p+1) + \sqrt{(p+1)^2+4}}{2} \right)^n$$

with

$$\alpha = \frac{1}{2} - \frac{p+1}{2\sqrt{p+1)^2+4}} \qquad \text{and} \qquad \beta = \frac{1}{2} + \frac{p+1}{2\sqrt{p+1)^2+4}}\,.$$

Corollary 6.3 *For $p = 0$, we have*

$$Z(C_n) = \frac{\sqrt{5}-1}{2\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n + \frac{\sqrt{5}+1}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n .$$

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