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Sub-Compatible Maps, Weakly Commuting Maps and Common Fixed Points in Cone Metric Spaces

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Abstract: The purpose of this paper is to obtain some common fixed point theorems under weaker conditions such as sub compatible mappings and weakly commuting with respect g in the setting of non - normal cone metric space.

Keywords: Sub-compatible maps, weakly commuting mappings, fixed point.

1 Introduction

The concept of cone metric spaces (or abstract spaces) introduced initially by Huang and Zhang [3]. In this space they have replaced completely ordered set R by real Banach Space E. Huang and Zhang proved Banach fixed point theorem of a complete metric space in complete cone metric space. For the fundamental importance of cone metric space which has bigger domain than of metric spaces. We define the following:

Definition 1 *Let E be a real Banach space and P subset of E. P is called a cone if and only if:*

1. *P is closed, nonempty, and $P \neq \{0\}$;*

2. *$a,b \in R$, $a,b \geq 0$, $x,y \in P \Rightarrow (ax+by) \in P$;*

3. *$x \in P$ and $-x \in P \Rightarrow x = 0$.*

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Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, $intP$ denotes the interior of P.
The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\| x \| \leq K \| y \|$. The least positive number satisfying above is called the normal constant of P.
The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is sequence such that $x_1 \leq x_2 \leq ... \leq x_n \leq y$ for some $y \in E$, then there is $x \in E$ such that $\| x_n - x \| \to 0$ ($n \to \infty$).
Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone. In the following we always suppose E is a Banach space, P is a cone in E with $intP \neq \phi$ and $\leq$ is partial ordering with respect to P.

Definition 2 *Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies:*

1. $0 < d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$;

2. $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

3. $d(x, y) \leq d(x, z) + d(y, z)$ *for all* $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space. It is obvious that cone metric spaces generalize metric spaces.

Example 1 *Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subset R^2$, $X = R$ and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \leq 0$ is a constant. Then (X, d) is a cone metric space.*

Definition 3 *Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 \ll c$ there is N such that for all $n > N$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and x_n converges to x, and x is the limit of x_n. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ ($n \to \infty$).*

Lemma 1 *Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ ($n \to \infty$).*

Lemma 2 *Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ ($n, m \to \infty$).*

Definition 4 *Let (X,d) be a cone metric space, $\{x_n\}$ be a sequence in X. If for any $c \in E$ with $0 \ll c$, there is N such that for all $n,m > N$, $d(x_n,x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.*

Corollary 1 *(see e.g., [7] - without proof).*

1. *If $a \leq b$ and $b \ll c$, then $a \ll c$.*
   Indeed, $c-a = (c-b)+(b-a) \geq c-b$ implies $[-(c-a),c-a] \supseteq [-(c-a),c-b]$.
2. *If $a \ll b$ and $b \ll c$, then $a \ll c$.*
   Indeed, $c-a = (c-b)+(b-a) > c-b$ implies $[-(c-a),c-a] \supset [-(c-a),c-b]$.
3. *If $0 \ll c$ for each $c \in intP$ then $c = 0$.*

In 1976 Jungck [4] generalized the Banach fixed point theorem for a pair of two commuting self-maps in complete metric space satisfying the following inequality:
$d(fx, fy) \leq d(gx, gy)$ for all $x,y \in X$, $\alpha \in [0,1)$.
After Jungck [5] and Sessa [8] weaken the concept of commuting map by weakly commuting maps. In 1986 Jungck [5] and in 1993 Jungck, Murthy and Cho [6] introduced the concepts of compatible and compatible maps of type (A) respectively in metric spaces by concrete example. It has been shown that both definitions are independent in nature (see [6]).
Bouhadjera and Godet-thobie [1] weaken the concept of weak compatible maps and occasionally weakly compatible respectively and define Sub-compatible maps in metric spaces. Here we shall extend the concept of sub-compatible pair of maps in cone metric spaces.

Definition 5 *Let (X,d) be a cone metric space. Let f and g be two self-maps of a cone metric space (X,d), then f and g are said to be sub-compatible maps, if and only if there exists a sequence $\{x_n\}$ in X such that $d(fx_n,z) = d(gx_n,z) \ll c$, for some $z \in X$ and $d(fgx_n, gfx_n) \ll c$ with $0 \ll c$, $c \in E$.*

2 Common Fixed Points under Sub-compatible Maps

Theorem 1 *Let (X,d) be a cone metric space with cone P having non - empty interior. Suppose that the mapping $f,g : X \to X$ satisfy*

$$d(f(x), f(y)) \leq \alpha d(f(x), g(x)) + \beta d(f(y), g(y)) + \gamma d(g(x), g(y)) \tag{1}$$

for all $x,y \in X$, where $\alpha, \beta, \gamma \in [0,1)$ and $\alpha + \beta + \gamma < 1$. If the range of g and $g(X)$ is a complete subspace of X then f and g have a unique common fixed point, provided f and g are sub-compatible maps.

Proof Let x_0 be an arbitrary point in X. Choose a point x_1 in X such that $f(x_0) = g(x_1)$. This can be done, since the range of g contains the range of f. Continuing this process, having chosen x_n in X, we obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1})$. Then from condition (1), we have

$$\begin{aligned} d(g(x_{n+1}),g(x_n)) &= d(f(x_n),f(x_{n-1})) \\ &\leq \alpha\, d(f(x_n),g(x_n)) + \beta\, d(f(x_{n-1}),g(x_{n-1})) + \gamma\, d(g(x_n),g(x_{n-1})) \\ &\leq \alpha\, d(g(x_{n+1}),g(x_n)) + \beta\, d(g(x_n),g(x_{n-1})) + \gamma\, d(g(x_n),g(x_{n-1})). \end{aligned}$$

We find

$$d(g(x_{n+1}),g(x_n)) \leq \frac{\beta+\gamma}{1-\alpha}\, d(g(x_n),g(x_{n-1})).$$

Consequently,

$$d(g(x_{n+1}),g(x_n)) \leq (\frac{\beta+\gamma}{1-\alpha})^n\, d(g(x_1),g(x_0)) = h^n\, d(g(x_1),g(x_0)),$$

where $\frac{\beta+\gamma}{1-\alpha} = h \in [0,1)$.

Now for $n > m \in N$, we have

$$\begin{aligned} d(g(x_n),g(x_m)) &\leq d(g(x_n),g(x_{n-1})) + d(g(x_{n-1}),g(x_{n-2})) + \ldots + d(g(x_{m+1}),g(x_m)) \\ &\leq (h^{n-1} + h^{n-2} + \ldots + h^m)d(g(x_1),g(x_0)) \\ &= h^m(h^{n-m-1} + h^{n-m-2} + \ldots + 1)d(g(x_1),g(x_0)) \\ &< h^m(1 + h + h^2 + \ldots)d(g(x_1),g(x_0)) \\ &= \tfrac{h^m}{1-h}\, d(g(x_1),g(x_0)). \end{aligned}$$

$\Rightarrow d(g(x_n),g(x_m)) \leq \frac{h^m}{1-h}\, d(g(x_1),g(x_0)) \to 0$ as $m \to \infty$.

So $d(g(x_n),g(x_m)) \leq 0$ as $m,n \to \infty$ and $0 \ll c$ be given.

Hence, by corollary (8) we get $d(g(x_n),g(x_m)) \ll c$.
Hence $\{g(x_n)\}$ is a Cauchy sequence.

Since $g(X)$ is a complete subspace of X then there exist $z \in g(X) \subset f(X)$ such that $g(x_n) \to z$ and also $f(x_n) \to z$ as $n \to \infty$.

Since f and g are sub - compatible maps so we have

$d(fgx_n, gfx_n) \ll c \Rightarrow fgx_n = gfx_n$. We obtain $f(z) = g(z)$ as $n \to \infty$.

Now remain to show that z is common fixed point of f and g.
If $z \neq f(z)$, We have

$$
\begin{aligned}
0 < d(f(z), z) \\
&\leq \alpha\, d(f(z), g(z)) + \beta\, d(f(z), g(z)) + \gamma\, d(g(z), g(z)) \\
&= \gamma\, d(f(z), f(z))
\end{aligned}
$$

this is a contradiction and so $f(z) = g(z) = z$. Then z is a common fixed point for the mappings f and g. The uniqueness follows from the contraction condition (1). If z' is another common fixed point. Then, we have

$$
\begin{aligned}
d(z', f(z)) &= d(f(z'), f(z)) \\
&\leq \alpha.\, d(f(z'), g(z')) + \beta.\, d(f(z), g(z)) + \gamma.\, d(g(z'), g(z)) \\
&= \gamma.\, d(f(z'), f(z)) \\
&\Rightarrow (1 - \gamma) .\, d(f(z'), f(z)) \leq 0
\end{aligned}
$$

and this implies that $f(z') = f(z)$ that is $z' = z$.
This completes the proof of the Theorem 12.

□

Corollary 2 *Let (X,d) be a cone metric space with cone P having non - empty interior. Suppose that the mapping $f,g : X \to X$ such that $f(X) \subset g(X)$ satisfying the following condition*

$$d(f(x), f(y)) \leq \alpha d(g(x), g(y)) \tag{2}$$

for all $x,y \in X$, where $\alpha \in [0,1)$.
If the range of g and $g(X)$ is a complete subspace of X then f and g have a unique common fixed point, provided f and g are sub-compatible maps.

3 Common Fixed Points under f,g and h weakly Commuting maps

Definition 6 *Let f,g and h are self maps of a cone metric space (X,d) are said to be weakly commuting with respect g iff*

$$d(hfhx, ghy) \leq d(fhhx, ghy)$$

for all $x \in X$.

Theorem 2 *Let (X,d) be a complete cone metric space with cone P having non - empty interior such that $d(x,y) \in IntP$, for all x, $y \in X$ with $x \neq y$. Let $f,g,h : X \longrightarrow X$ such that*

$$\Psi(d(fhx, ghy)) \leq \Psi(d(x,y)) - \varphi(d(x,y)) \tag{3}$$

for all x, $y \in X$ where $\Psi : P \to P$ and $\varphi : IntP \cup \{0\} \to IntP \cup \{0\}$ are continuous functions with the following properties:

1. *Ψ is monotonic increasing;*
2. *$\Psi(t) = 0 = \varphi(t)$ iff t = o;*
3. *either $\varphi(t) \leq$d(x, y) or d(x, y) $\leq \varphi(t)$.*

for $t \in IntP \cup \{0\}$ and x, $y \in X$.
If f, g and h are weakly commuting pair of maps with respect to g; then f, g and h have a unique common fixed point in X.

Proof Let $x_0 \in X$ be an arbitrary point and we define $x_1 = fh(x_0)$ and $x_2 = gh(x_1)$, inductively we shall define:
$x_{2n+1} = fh(x_{2n})$ and $x_{2n+2} = gh(x_{2n+1})$.
Let $d_n = d(x_n, x_{n+1})$. If $x_{2n} = x_{2n+1}$, then $\{x_n\}$ is a Cauchy sequence. If $x_{2n} \neq x_{2n+1}$, then from (3), we have

$$\begin{aligned}\Psi(d_{2n+1}) &= \Psi(d(x_{2n+1}, x_{2n+2}) \\ &= \Psi(d(fhx_{2n}, ghx_{2n+1}) \\ &= \Psi(d(x_{2n}, x_{2n+1})) - \varphi(d(x_{2n}, x_{2n+1})) \text{ for } n \in N\end{aligned}$$

i.e.

$$\Psi(d_{2n+1}) \leq \Psi(d_{2n}) - \varphi(d_{2n}). \tag{4}$$

By using property of φ $\Psi(d_{2n+1}) \leq \Psi(d_{2n})$,
which implies $d_{2n+1} \leq d_{2n}$ (by monotone property of φ).
Therefore $\{d_n\}$ is monotonically decreasing. Hence by *Lemma* (3.1) of Choudhary and Metiya [2] there exists an $\lambda \in P$ with either $\lambda = 0$ or $\lambda \in IntP$, such that

$$d_n \to \lambda \text{ as } n \to \infty. \tag{5}$$

Taking the limit as n $\to \infty$ in (4) by using (5), we have

$$\Psi(\lambda) \leq \Psi(\lambda) - \varphi(\lambda)$$

a contradiction otherwise $\lambda = 0$.
Therefore

$$d_n = d(x_n, x_{n+1}) \to 0 \; as \; n \to \infty. \tag{6}$$

Now, we shall show that $\{x_n\}$ be a Cauchy sequence. If not, then there exists c $\in$ E with $0 \ll c$, such that for every $n_0 \in \mathbb{N}$, there exists n, m with $n > m \geq n_0$ such that $d(x_n, x_m) \ll \varphi(c)$. Hence, by property of φ in (iv) $\varphi(c) \ll d(x_n, x_m)$.
Therefore there exists sequences, $\{m(k)\}$ and $\{n(k)\}$ in N such that for all positive integer k, $n(k) > m(k) > k$ and $d(x_{m(k)}, x_{n(k)}) \geq \varphi(c)$.
Suppose that $n(k)$ is the smallest such positive integer, we have

$$d(x_{m(k)}, x_{n(k)}) \geq \varphi(c)$$

and

$$d(x_{m(k)}, x_{n(k)-1}) \leq \varphi(c).$$

Now,
$\varphi(c) \leq d(x_{m(k)}, x_{n(k)})$
$\leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)})$.
Thus

$$\varphi(c) \leq \varphi(c) + d(x_{n(k)-1}, x_{n(k)}).$$

Taking $k \to \infty$, in the above inequality and using (4)

$$lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \varphi(c). \tag{7}$$

Again

$$d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)}).$$

and

$$d(x_{m(k)+1}, x_{n(k)+1}) \leq d(x_{m(k)+1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1}).$$

Taking $k \to \infty$ in the above inequality, by using (7) and (6), we have

$$lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \varphi(c). \tag{8}$$

Putting x = $x_{m(k)}$ and y = $y_{n(k)}$ in (3), we have

$\Psi(d(fhx_{m(k)}, ghx_{n(k)})) \leq \Psi(d(x_{m(k)}, x_{n(k)})) - \varphi(d(x_{m(k)}, x_{n(k)}))$

i.e.
$\Psi(d(x_{m(k)+1}, x_{n(k)+1})) \leq \Psi(d(x_{m(k)}, x_{n(k)})) - \varphi(d(x_{m(k)}, x_{n(k)}))$
letting $k \to \infty$ in the above inequality and using (7) and (8), and the continuity of $\Psi and\ \varphi$, we have

$$\Psi(\varphi(c)) \leq \Psi(\varphi(c)) - \varphi(\varphi(c))$$

which is a contradiction.
Therefore, $\{x_n\}$ is a Cauchy sequence in X. Since X is a complete cone metric space, there exists a point $\xi \in$ X such that

$$x_n \to \xi \ as\ n \to \infty. \tag{9}$$

Now, we shall show that $fh\xi = \xi$. From (3), we have

$\Psi(d(fh\xi, ghx_{2n+1})) \leq \Psi(d(\xi, x_{2n+1})) - \varphi(d(\xi, x_{2n+1}))$
or
$\Psi(d(fh\xi, x_{2n+2})) \leq \Psi(d(\xi, x_{2n+1})) - \varphi(d(\xi, x_{2n+1}))$.

Taking $x_n \to \infty$ in the above inequality, using (8) and the property of Ψ and φ, we have

$\Psi(d(fh\xi, \xi)) \ll c$ which implies $d(fh\xi, \xi)) \leq 0$ i.e. $fh\xi = \xi$.
Similarly, we can show that $gh\xi = \xi$ and we shall obtain ξ is a common fixed point of fh and gh i.e. $fh\xi = \xi = gh\xi$.
Since f, g and h is a weakly commuting pair of maps with respect to g, then

$$\begin{aligned}\Psi(d(h\xi, \xi)) &= \Psi(d(hfh\xi, gh\xi)) \\ &\leq \Psi(d(fhh\xi, gh\xi)) \ \ \text{(since } \Psi \text{ is monotone increasing)} \\ &\leq \Psi(d(h\xi, \xi)) - \varphi(d(h\xi, \xi))\end{aligned}$$

which is a contradiction. Thus, $\Psi(d(h\xi, \xi)) = 0 \Rightarrow h\xi = \xi = fh\xi = gh\xi$ i.e.$h\xi = \xi = f\xi = g\xi$.
Hence, ξ is a common fixed point of f, g and h.

For uniqueness of ξ once again we shall use inequality (3).
Hence, ξ is a unique common fixed point of f, g and h.
This completes the proof of Theorem 15.

□

Corollary 3 *Let (X,d) be a complete cone metric space with cone P having non-empty interior such that $d(x,y) \in IntP$, for all $x,y \in X$ with $x \neq y$. Let $f,g,h : X \longrightarrow X$ such that*

$$\Psi(d(fhx,ghy)) \leq \Psi(d(x,fhx)+d(y,ghx)) - \varphi(d(x,y)) \tag{10}$$

for all $x,y \in X$ where $\Psi : P \to P$ and $\varphi : IntP \cup \{0\} \to IntP \cup \{0\}$ are continuous functions with the following properties:

1. *Ψ is monotonic increasing;*
2. *$\Psi(t) = 0 = \varphi(t)$ iff $t = o$;*
3. *either $\varphi(t) \leq d(x,y)$ or $d(x,y) \leq \varphi(t)$;*

for $t \in IntP \cup \{0\}$ and $x,y \in X$.
If f, g and h are weakly commuting pair of maps with respect to g; then f, g and h have a unique common fixed point in X.

Corollary 4 *Let (X,d) be a complete cone metric space with cone P having non - empty interior such that $d(x,y) \in IntP$, for all $x,y \in X$ with $x \neq y$. Let $f,g,h : X \longrightarrow X$ such that*

$$\Psi(d(fhx,ghy)) \leq \Psi(\frac{1}{2}[d(x,ghy)+d(y,fhx)] - \varphi(d(x,y) \tag{11}$$

for all $x,y \in X$ where $\Psi : P \to P$ and $\varphi : IntP \cup \{0\} \to IntP \cup \{0\}$ are continuous functions with the following properties:

1. *Ψ is monotonic increasing;*
2. *$\Psi(t) = 0 = \varphi(t)$ iff $t = o$;*
3. *either $\varphi(t) \leq d(x,y)$ or $d(x,y) \leq \varphi(t)$;*

for $t \in IntP \cup \{0\}$ and $x,y \in X$.
If f, g and h are weakly commuting pair of maps with respect to g; then f, g and h have a unique common fixed point in X.

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