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Best Proximity Points of Local Contractive Mappings on Metric Spaces Endowed with Binary Relation

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Abstract: The aim of this paper is to present best proximity point results of $(\alpha-\eta,\psi)-$ proximal mappings satisfying local contractive conditions on a closed ball in the framework of complete metric spaces. An example is also presented to validate the result proved herein. As an application of our results, we prove existence of best proximity points of locally contractive mappings in the frame work of metric spaces endowed with binary relation. Our results extend and generalize various comparable results in the existing literature.

Keywords: Best proximity point; Non-self-mapping; $(\alpha-\eta,\psi)$-proximal contraction; closed ball.

1 Introduction and Preliminaries

Let A and B be two nonempty subsets of a metric space (X,d) and $T : A \to B$. An point $x \in A$ is said to be a fixed point of T provided that $Tx = x$. A point x^* in A where $\inf\{d(x,Tx^*) : x \in A\}$ is attained, that is, x^* is best approximation to $Tx^* \in B$ in A. Such a point is called an approximate fixed point of T.

Clearly, $T(A) \cap A \neq \emptyset$ is a necessary but not sufficient condition for the existence of a fixed point of T. If $T(A) \cap A = \emptyset$, then $d(x,Tx) > 0$ for all $x \in A$ and hence an operator equation $Tx = x$ does not admit a solution. In such situations, it is a reasonable demand to settle down with a point x^* in A which is closest to Tx^* in B. Thus instead of having $d(x^*,Tx^*) = 0$, one finds a point x^* in A such that $d(x^*,Tx^*) \leq d(x,Tx^*)$ holds for all x in A. Such point is called a best approximate point of T or approximate fixed point of T. The study of conditions that assure existence and uniqueness of approximate fixed point of a mapping T is an active area of research.

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Suppose that $d(A,B) = \inf(\{d(a,b) : a \in A, b \in B\})$ is the measure of a distance between two sets A and B. A point x^* is called a best proximity point of T if $d(x^*, Tx^*) = d(A,B)$. Thus a best proximity point problem defined by a mapping T and a pair of sets (A,B) is to find a point x^* in A such that $d(x^*, Tx^*) = d(A,B)$. As $d(x,Tx) \geq d(A,B)$ holds for all $x \in A$, so the global minimum of the mapping $x \to d(x,Tx)$ is attained at a best proximity point. If we take $A = B$, then a best proximity point problem reduces to fixed point problem. From this perspective, best proximity point problem can be viewed as a natural generalization of fixed point problem. The aim of best proximity point theory is to study sufficient conditions that assure the existence of best proximity points of mappings satisfying certain contractive conditions on its domain equipped with some distance structure.

For more results in this direction, we refer to [1], [4] [7, 13, 14, 15], [24] and references therein.

Over the past years, fixed point theory has been generalized in different directions by several mathematicians. Samet et al. [22] introduced the concept of (α, ψ) - contractive type mappings and obtained fixed point of such mappings in complete metric spaces. Karapinar et al. [8] modified the notion of (α, ψ) - contractive type mappings. Recently, Salimi et al. [19] modified the concept of $(\alpha, \psi)-$ contractive mappings further and obtained fixed point results. Hussain et al. [12] extended the concept of α-admissible mappings and obtained some interesting fixed point results. Subsequently, Abdeljawad [3] introduced pairs of $\alpha-$admissible mappings satisfying new contractive conditions different from those in [12, 22] and proved fixed and common fixed point results. Mohammadi et al. [9] introduced the notion of $(\alpha, \phi)-$contractive mappings and showed that fixed point results for such mappings are potential generalization of comparable existing results.

Fixed points results of mappings satisfying certain contractive conditions on the entire domain has been at the centre of rigorous research activity and it has a wide range of applications in different areas such as nonlinear and adaptive control systems, parameterize estimation problems, fractal image decoding, computing magneto static fields in a nonlinear medium, and convergence of recurrent networks. From the application point of view the situation is not yet completely satisfactory because it frequently happens that a mapping T is a contraction not on the entire space X. Arshad et al. [2] established fixed point results of a pair of contractive dominated mappings on a closed ball in an ordered complete dislocated metric space. Hussain et al. [11] introduced the concept of an α-admissible mappings with respect to η and modified (α, ψ)-contractive condition for a pair of mappings and established common fixed point results of four mappings on a closed ball in complete dislocated metric space.

Jleli et al. [6] obtained best proximity point results of (α, ψ)- proximal contractive type mappings in complete metric space. For more work in this direction, we refer to [5, 18], [20, 21], [16] and [23]. In this paper, we obtain best proximity point results of $(\alpha - \eta, \psi)-$ proximal local contractive mappings on a closed ball in complete metric spaces. We also prove existence of best proximity points of locally contractive mappings in the setup of metric spaces endowed with a binary relation. Our results extend, unify and generalize various comparable results in [6, 13, 14].

In the sequel the letter $\mathbb{N}$ will denote the set of all natural numbers. Following definitions, notations and results will also be needed in the sequel.

Let (X,d) be a metric space, A and B are nonempty subsets of X. For $x_0 \in X$ and $\varepsilon > 0$, the set $\overline{B(x_0,\varepsilon)} = \{y \in X : d(x_0,y) \leq \varepsilon\}$ is a closed ball in X.

Define: $\Psi = \{\psi : [0,\infty) \to [0,\infty)$ such that ψ is nondecreasing and $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0\}$.

Lemma 1 *([19]). If $\psi \in \Psi$, then $\psi(t) < t$ for all $t > 0$.*

Definition 1 *([22]). Let $\alpha : X \times X \to [0,\infty)$. A selfmap T on X is called α-admissible if for any $x,y \in X$ with $\alpha(x,y) \geq 1$ we have $\alpha(Tx,Ty) \geq 1$.*

Definition 2 *([22]). A mapping $T : X \to X$ is an $(\alpha, \psi)-$ contractive mapping if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that*

$$\alpha(x,y)d(Tx,Ty) \leq \psi(d(x,y))$$

holds for all $x,y \in X$.

Definition 3 *([19]). Let $T : X \to X$ and $\alpha,\eta : X \times X \to [0,\infty)$. We say that T is α-admissible mapping with respect to η if for any $x,y \in X$, $\alpha(x,y) \geq \eta(x,y)$ implies that $\alpha(Tx,Ty) \geq \eta(Tx,Ty)$.*

If $\eta(x,y) = 1$, then above definition reduces to definition 2. If in above definition, $\alpha(x,y) = 1$ then T is called an η-subadmissible mapping.

Suppose that

$$\begin{aligned} A_0 &:= \{a \in A : d(a,b) = d(A,B) \text{ for some } b \in B\} \text{ and} \\ B_0 &:= \{b \in B : d(a,b) = d(A,B) \text{ for some } a \in A\} \ . \end{aligned}$$

Definition 4 *([17]). Suppose that $A_0 \neq \emptyset$. A pair (A,B) is said to have the P-property if following holds:*

$$\begin{cases} d(x_1,y_1) = d(A,B) \\ d(x_2,y_2) = d(A,B) \end{cases} \text{ implies that } d(x_1,x_2) = d(y_1,y_2)$$

where $x_1,x_2 \in A_0$ and $y_1,y_2 \in B_0$.

Definition 5 *([6]). Let $\alpha : A \times A \to [0,\infty)$. A mapping $T : A \to B$ is α-proximal admissible if for any $x_1, x_2, u_1, u_2 \in A$,*

$$\begin{cases} \alpha(x_1,x_2) \geq 1 \\ d(u_1,Tx_1) = d(A,B) \\ d(u_2,Tx_2) = d(A,B) \end{cases} \text{implies that } \alpha(u_1,u_2) \geq 1,$$

Clearly, if $A = B$ and T is α-proximal admissible then T is α-admissible.

Definition 6 *Let $\alpha, \eta : A \times A \to [0,\infty)$. A mapping $T : A \to B$ is $(\alpha$-$\eta)$-proximal admissible if for any $x_1, x_2, u_1, u_2 \in A$,*

$$\begin{cases} \alpha(x_1,x_2) \geq \eta(x_1,x_2) \\ d(u_1,Tx_1) = d(A,B) \\ d(u_2,Tx_2) = d(A,B) \end{cases} \text{implies that } \alpha(u_1,u_2) \geq \eta(u_1,u_2),$$

Note that, if $A = B$ and T is $(\alpha$-$\eta)$-proximal admissible then T is α-admissible with respect to η.

Definition 7 *([6]). Let $\alpha : A \times A \to [0,\infty)$ and $\psi \in \Psi$. A mapping $T : A \to B$ is said to be an (α, ψ)- proximal contraction if for any $x, y \in A$, the following condition hold:*

$$\alpha(x,y)d(Tx,Ty) \leq \psi(d(x,y)).$$

Definition 8 *Let $\alpha, \eta : A \times A \to [0,\infty)$ and $\psi \in \Psi$. A mapping $T : A \to B$ is said to be an $(\alpha - \eta, \psi)$- proximal contraction if for any $x, y \in A$,*

$$\alpha(x,y) \geq \eta(x,y) \text{ implies that } d(Tx,Ty) \leq \psi(d(x,y)).$$

Note that if we take $\eta(x,y) = \frac{1}{2}d(x,Tx)$ and $\alpha(x,y) = d(x,y)$, then $(\alpha - \eta, \psi)$- proximal contraction mapping becomes Suzuki type mapping in [10].

2 Main results

We start with the following result.

Theorem 1 *Let A and B be nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty and $T : A \to B$ an $(\alpha$-$\eta)$-proximal admissible mapping. Suppose that for any $x, y \in \overline{B(x_0,r)}$ with $\alpha(x,y) \geq \eta(x,y)$ we have*

$$d(Tx,Ty) \leq \psi(d(x,y)), \tag{1}$$

where $\psi \in \Psi$, $x_0 \in A$ *and*

$$\sum_{i=0}^{j} \psi^i(d(x_0,Tx_0)) \leq r \textit{ for all } j \in \mathbb{N} \cup \{0\}. \tag{2}$$

If the following conditions hold:

(i) $T(A_0) \subseteq B_0$ and the pair (A,B) satisfies the P-property;

(ii) There exist an element $x_1 \in A_0$ such that $d(x_1,Tx_0) = d(A,B)$ and $\alpha(x_0,x_1) \geq \eta(x_0,x_1)$;

(iii) T is a continuous and $(\alpha\text{-}\eta)$-proximal admissible.

Then, there exists an element $x^* \in \overline{B(x_0,r)}$ such that $d(x^*,Tx^*) = d(A,B)$.

Proof. From condition (ii), there exists an element x_1 in A_0 such that $d(x_1,Tx_0) = d(A,B)$ and $\alpha(x_0,x_1) \geq \eta(x_0,x_1)$. As $T(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that $d(x_2,Tx_1) = d(A,B)$. Since T is an $(\alpha\text{-}\eta)$-proximal admissible, we have $\alpha(x_1,x_2) \geq \eta(x_1,x_2)$. Thus, we have $d(x_2,Tx_1) = d(A,B)$ and $\alpha(x_1,x_2) \geq \eta(x_1,x_2)$. Similarly we can choose $x_3 \in A_0$ such that $d(x_3,Tx_2) = d(A,B)$, and hence $\alpha(x_2,x_3) \geq \eta(x_2,x_3)$. Continuing this way, we can obtain a sequence $\{x_n\} \subset A_0$ such that $d(x_{n+1},Tx_n) = d(A,B)$ and it satisfies:

$$\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1}), \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{3}$$

Since (A,B) satisfies the P-property, we have

$$d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n), \text{ for all } n \in \mathbb{N}. \tag{4}$$

Now we show that $x_n \in \overline{B(x_0,r)}$ for all $n \in \mathbb{N}$. By (2), we have $d(x_0,Tx_0)) \leq r$ and hence $x_1 \in \overline{B(x_0,r)}$. Let $x_2,\cdots,x_j \in \overline{B(x_0,r)}$ for some $j \in \mathbb{N}$. From the fact that $\alpha(x_{i-1},x_i) \geq \eta(x_{i-1},x_{i-1})$ and T is an $(\alpha-\eta,\psi)$-proximal contraction, it follows that

$$d(Tx_{i-1},Tx_i) \leq \psi(d(x_{i-1},x_i)) \tag{5}$$

$\forall i \in \mathbb{N}$ and hence

$$d(x_i,x_{i+1}) \leq \psi^i(d(x_0,x_1)). \tag{6}$$

Note that

$$\begin{aligned} d(x_0,x_{j+1}) &= d(x_0,x_1)+d(x_1,x_2)+d(x_2,x_3)+...+d(x_j,x_{j+1}) \\ &\leq \sum_{i=0}^{j} \psi^i(d(x_0,x_1)) \leq r \end{aligned}$$

which implies that $x_{j+1} \in \overline{B(x_0,r)}$ and hence $x_n \in \overline{B(x_0,r)}$ for all $n \in \mathbb{N}$.

If for some positive integer k, we have $x_k = x_{k+1}$. Then, we have $d(x_k, Tx_k) = d(x_{k+1}, Tx_k) = d(A,B)$, that is, x_k is a best proximity point of T. Assume that $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Now, we prove that $\{x_n\}$ is a Cauchy sequence. Fix $\varepsilon > 0$. As $\sum_{n=1}^{\infty} \psi^n(d(x_1,x_0)) < \infty$, there exists some positive integer $N \in \mathbb{N}$ such that $\sum_{n \geq N} \psi^n(d(x_1,x_0)) < \varepsilon$. So for $m,n \in \mathbb{N}$ with $m > n > N$,

$$\begin{aligned} d(x_n,x_m) &\leq \sum_{k=n}^{m-1} d(x_k,x_{k+1}) \leq \sum_{k=n}^{m-1} \psi^k(d(x_1,x_0)) \\ &\leq \sum_{n\geq N} \psi^n(d(x_1,x_0)) < \varepsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in $(\overline{B(x_0,r)},d)$. Since (X,d) is complete, there exists $x^* \in \overline{B(x_0,r)}$ such that $x_n \to x^*$ as $n \to \infty$. The continuity of T implies that $Tx_n \to Tx^*$ as $n \to \infty$ and hence $d(A,B) = d(x_{n+1}, Tx_n) \to d(x^*, Tx^*)$ as $n \to \infty$. Therefore, $d(x^*, Tx^*) = d(A,B)$. ■

In the following Theorem, the assumption of continuity is replaced with following suitable condition:

(H) If $\{x_n\}$ is a sequence in A such that $\alpha(x_n,x_{n+1}) \geq 1$ for all n and $x_n \to x^* \in A_0$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x^*) \geq \eta(x_{n(k)},x^*)$ for all k.

Theorem 2 *Let A and B be nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty and $T : A \to B$ an (α-η)-proximal admissible. Suppose that for any $x,y \in \overline{B(x_0,r)}$ with $\alpha(x,y) \geq \eta(x,y)$ we have*

$$d(Tx,Ty) \leq \psi(d(x,y)),$$

where $\psi \in \Psi$ and

$$\sum_{i=0}^{j} \psi^i(d(x_0,Tx_0)) \leq r, \text{ for all } j \in \mathbb{N} \cup \{0\}.$$

If the following assertions hold:

(i) $T(A_0) \subseteq B_0$ and (A,B) satisfies the P-property;
(ii) There exist elements $x_1 \in A_0$ such that $d(x_1,Tx_0) = d(A,B)$ and $\alpha(x_0,x_1) \geq \eta(x_0,x_1)$;
(iii) Condition (H) holds.
Then, there exists an element $x^* \in \overline{B(x_0,r)}$ such that $d(x^*,Tx^*) = d(A,B)$.

Proof. By Theorem 10 it follows that there exists a Cauchy sequence $\{x_n\}$ is a sequence in A such that $d(x_{n+1},Tx_n) = d(A,B)$ and it satisfies

$$\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1}),$$

for all $n \in \mathbb{N} \cup \{0\}$ holds and $x_n \to x^* \in \overline{B(x_0,r)}$ as $n \to \infty$. From the condition (H), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x^*) \geq \eta(x_{n(k)},x^*)$ for all k. We claim that

$$Tx_{n(k)} \to Tx^* \text{as } k \to \infty. \tag{7}$$

As T is an $(\alpha-\eta,\psi)$-proximal contraction and $\alpha(x_{n(k)},x^*) \geq \eta(x_{n(k)},x^*)$, so we have

$$d(Tx_{n(k)},Tx^*) \leq \psi\left(d(x_{n(k)},x^*)\right),\ \forall k.$$

The claim follows immediately on taking limit as $k \to \infty$ on both sides of above inequality. That is,

$$d(A,B) = d(x_{n(k)+1},Tx_{n(k)}) \to d(x^*,Tx^*) \text{ as } k \to \infty.$$

and hence $d(x^*,Tx^*) = d(A,B)$. ■

If we take $\eta(x,y) = 1$ for any $x,y \in X$ in Theorem 10, we obtain the following result.

Corollary 1 *Let A and B be nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty and $T : A \to B$ is continuous α-proximal admissible mapping. Suppose that for any $x,y \in \overline{B(x_0,r)}$ with $\alpha(x,y) \geq 1$ we have*

$$d(Tx,Ty) \leq \psi(d(x,y)),$$

where $\psi \in \Psi$, $x_0 \in A$ and

$$\sum_{i=0}^{j} \psi^i(d(x_0,Tx_0)) \leq r \text{ for all } j \in \mathbb{N} \cup \{0\}.$$

If the following conditions hold:

(i) $T(A_0) \subseteq B_0$ and the pair (A,B) satisfies the P-property;
(ii) There exist an element $x_1 \in A_0$ such that $d(x_1,Tx_0) = d(A,B)$ and $\alpha(x_0,x_1) \geq 1$.
Then, there exists an element $x^* \in \overline{B(x_0,r)}$ such that $d(x^*,Tx^*) = d(A,B)$.

If $\eta(x,y) = 1$ for all $x,y \in X$ in Theorem 11 we obtain following result.

Corollary 2 *Let A and B be nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty and $T : A \to B$ an α-proximal admissible. Suppose that for any $x,y \in \overline{B(x_0,r)}$ with $\alpha(x,y) \geq 1$ we have*

$$d(Tx,Ty) \leq \psi(d(x,y)),$$

where $\psi \in \Psi$ and

$$\sum_{i=0}^{j} \psi^i(d(x_0,Tx_0)) \leq r, \text{ for all } j \in \mathbb{N} \cup \{0\}.$$

If the following assertions hold:

(i) $T(A_0) \subseteq B_0$ and (A,B) satisfies the P-property;
(ii) There exist elements $x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A,B)$ and $\alpha(x_0,x_1) \geq 1$;
(iii) Condition (H) holds.
Then, there exists an element $x^* \in \overline{B(x_0,r)}$ such that $d(x^*, Tx^*) = d(A,B)$.

Example 1 *Let $X = \mathbb{R}$ and $|.|$ the usual metric on $\mathbb{R}$. Define the mapping $T : A \to B$ by*

$$Tx = \begin{cases} x+2 \text{ if } x \in [0,1) \\ 2 \text{ if } x = 1. \end{cases},$$

where $A = [0,1]$, $B = [2,3]$. Consider $x_0 = 1$, $r = 3$, $\psi(t) = \frac{t}{2}$ and

$$\alpha(x,y) = \begin{cases} 1 \text{ if } x,y \geq 0 \\ 0 \text{ otherwise.} \end{cases} \text{ and } \eta(x,y) = \frac{1}{2}.$$

Note that $\overline{B(x_0,r)} = [0,1]$,

$$\begin{aligned} A_0 &: = \{a \in A : d(a,b) = d(A,B) \text{ for some } b \in B\} = \{1\} \text{ and} \\ B_0 &: = \{b \in B : d(a,b) = d(A,B) \text{ for some } a \in A\} = \{2\}. \end{aligned}$$

Obviously $T(A_0) \subseteq B_0$ and $\alpha(1,1) \geq \eta(1,1)$. Note that

$$\begin{aligned} d(A,B) &= d(x_0,Tx_0) = d(1,T1) = |1-3| = 2 \text{ and} \\ \sum_{i=0}^{i} \psi^i(d(A,B)) &= 2\sum_{i=0}^{n} \frac{1}{3^i} < 3. \end{aligned}$$

Let $x,y \in [0,1]$. If

$$\begin{cases} \alpha(x,y) \geq \eta(x,y) \\ d(u,Tx) = d(A,B) = 1 \\ d(v,Ty) = d(A,B) = 1 \end{cases}$$

then we have

$$\begin{cases} x,y \in [0,1] \\ d(u,Tx) = 1 \\ d(v,Ty) = 1 \end{cases}$$

Hence $\alpha(u,v) \geq \eta(u,v)$ and T is $(\alpha$-$\eta)$-proximal admissible. If $x,y \in \overline{B(x_0,r)}$, then

$$\begin{aligned} 2|x-y| &\leq |x-y| \text{ or } |x-y| \leq \frac{|x-y|}{2}. \\ \text{or } |x+2-(y+2)| &\leq \psi(|x-y|) \text{ or } d(Tx,Ty) \leq \psi(d(x,y)). \end{aligned}$$

Hence T is $(\alpha - \eta, \psi)$- proximal contraction. Moreover, if $\{x_n\}$ is a sequence such that $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$, then $\{x_n\} \subseteq [0,1]$ and hence $x \in [0,1]$. Consequently, $\alpha(x_n,x) \geq \eta(x_n,x)$ for all $n \in \mathbb{N} \cup \{0\}$. Therefore all the conditions of Theorem 10 and Theorem 11 hold true. Hence T has a best proximity point.

3 Best Proximity point on closed ball endowed with binary relation

Let (X,d) be a metric space and $\Re$ a binary relation on X. Denote $\mathbf{S} = \Re \cup \Re^{-1}$, the symmetric relation associated with $\Re$. Note that

$$x,y \in X,\ x\mathbf{Sy} \Longleftrightarrow x\Re y \text{ or } y\Re x.$$

Definition 9 *([6]). A mapping $T : A \to B$ is called a comparative mapping if*

$$\begin{cases} x_1\mathbf{S}x_2 \\ d(u_1,Tx_1) = d(A,B) \\ d(u_2,Tx_2) = d(A,B) \end{cases} \text{implies that } u_1\mathbf{S}u_2,$$

for all $x_1,x_2,u_1,u_2 \in A$.

Now we state the following best proximity point result

Theorem 3 *Let A and B be two nonempty closed subsets of a complete metric space $\overline{(X,d)}$ such that A_0 is nonempty and $\Re$ a binary relation over X. Suppose that for any $x,y \in \overline{B(x_0,r)}$ with $x\mathbf{S}y$, we have*

$$d(Tx,Ty) \le \psi\left(d(x,y)\right), \tag{8}$$

where $\psi \in \Psi$ and

$$\sum_{i=0}^{j} \psi^i(d(x_0,Tx_0)) \le r, \text{ for all } j \in \mathbb{N}. \tag{9}$$

Suppose that $T : A \to B$ is continuous mapping satisfying the following conditions:

(i) $T(A_0) \subseteq B_0$ and (A,B) satisfies the P-property;
(ii) T is a proximal comparative mapping;
(iii) There exist an element $x_1 \in A_0$ such that $d(x_1,Tx_0) = d(A,B)$ and $x_0\mathbf{S}x_1$.
Then, there exists an element $x^* \in \overline{B(x_0,r)}$ such that $d\left(x^*,Tx^*\right) = d(A,B)$.

Proof. Let $\alpha : A \times A \to [0,\infty)$ be a mapping defined by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x\mathbf{S}y \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Suppose that

$$\begin{cases} \alpha(x_1,x_2) \ge 1 \\ d(u_1,Tx_1) = d(A,B) \\ d(u_2,Tx_2) = d(A,B) \end{cases} \text{implies that } \alpha(u_1,u_2) \ge 1,$$

for some $x_1, x_2, u_1, u_2 \in A$. By the definition of α we have

$$\begin{cases} x_1 \mathbf{S} x_2 \\ d(u_1, Tx_1) = d(A,B) \\ d(u_2, Tx_2) = d(A,B) \end{cases}$$

implies that $u_1 \mathbf{S} u_2$. Note that T is an $(\alpha - \eta, \psi)$-proximal contraction. Thus all the hypotheses of Theorem 10 are satisfied, and hence the result follows. ■

$(\hat{H})$ If $\{x_n\}$ is a sequence in X and $x \in X$ are such that $x_n \mathbf{S} x_{n+1}$ for all n and $x_n \to x^* \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \mathbf{S} x^*$ for all k.

Theorem 4 *Let A and B be nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty and $\mathfrak{R}$ a binary relation over X. Suppose that for any $x, y \in \overline{B(x_0, r)}$ with $x \mathbf{S} y$, we have*

$$d(Tx, Ty) \leq \psi(d(x,y)), \tag{11}$$

where $\psi \in \Psi$ and

$$\sum_{i=0}^{j} \psi^i(d(x_0, Tx_0)) \leq r, \text{ for all } j \in \mathbb{N}. \tag{12}$$

If the mapping $T : A \to B$ satisfies the following conditions:

(i) $T(A_0) \subseteq B_0$ and (A,B) satisfies the P-property;
(ii) T is a proximal comparative mapping;
(iii) There exist an element $x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A,B)$ and $x_0 \mathbf{S} x_1$;
(iv) $\hat{H}$ holds.
Then, there exists an element $x^* \in \overline{B(x_0, r)}$ such that $d(x^*, Tx^*) = d(A,B)$.

Proof. The result follows from Theorem 11, considering the mapping α given by (10) and by observing that condition $(\hat{H})$ implies condition (H). ■

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