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Non-normal cone metric and cone b-metric spaces and fixed point results

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Abstract: We show that most fixed point results obtained so far in cone metric spaces over solid non-normal cones can be easily reduced to the case of solid normal cones and, hence, their proofs can be made much simpler. Also, cone tvs-valued spaces over solid cones are not an essential generalization of cone metric spaces. These results are consequences of the simple fact that each solid cone in a topological vector space is in fact normal under a suitably defined norm. The proof follows by using the technique of Minkowski functional. As an application of these results, we prove an extension of the classical Nemytzki-Edelstein fixed point result to (tvs)-(b)-cone metric spaces over solid cones.

Keywords: Topological vector space; ordered normed space; cone metric space; b-metric space; tvs-cone b-metric space; Minkowski functional.

1 Introduction

The connection between cones and order relations in vector spaces is very well known. In particular, the usage of ordered normed spaces in Functional Analysis date back to 1940's (see [24, 25, 30, 31]). It seems that Kurepa ([26]) was the first to use ordered normed spaces as the codomain of a metric (see [28]). Later on, such "metric" spaces appeared occasionally under various names: K-metric spaces, abstract metric space, generalized metric spaces (see, e.g., [32]).

The spaces of this type were re-introduced in 2007 by Huang and Zhang [16] under the name of cone metric spaces. Among other things, they used the relation $\ll$ (mentioned already in [24]) which could be defined under the supposition that the underlying cone had a nonempty interior (such cones are usually called solid). Later, these definitions were

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extended for topological vector space-valued cone metric space (or tvs-cone metric space) in [4, 10, 21]. Afterwards, several authors obtained a lot of fixed point results in (tvs)-cone metric spaces.

Several authors showed (by various methods) that each cone metric space over a solid cone is metrizable (see, e.g., [2, 5, 10, 15, 20, 23]). However, this does not mean that all fixed point results can be reduced in this way to their standard metric counterparts. For example, it is still not known whether scalar and vector versions of the celebrated Caristi's fixed point result are equivalent (see [22]).

In the paper [16], the underlying cones were supposed to be normal (see the definition in next section). It was shown in [29] that such assumption is sometimes not necessary. Hence, further, a lot of authors obtained (common) fixed point results for non-normal solid cones (see a survey of these results until 2011 in [19]; a lot of papers appeared afterwards, too). However, the respective proofs were usually rather long and not direct extensions of the known proofs from the standard metric case.

In this paper, we show that most fixed point results obtained so far in cone metric spaces over solid non-normal cones can be easily reduced to the case of solid normal cones and, hence, their proofs can be made much simpler. Also, cone tvs-valued spaces over solid cones are not an essential generalization of cone metric spaces. These results are consequences of the simple fact that each solid cone in a *tvs* is in fact normal under a suitably defined norm (not always equivalent with the original one). The proof follows by using the technique of Minkowski functional, already used in [20]. As an application of these results, we prove an extension of the classical Nemytzki-Edelstein fixed point result [13, 27] to tvs-cone metric spaces over solid cones.

Continuing in this direction, we consider tvs-cone b-metric spaces as a generalization of b-metric spaces and prove that some of fixed point theorems in cone b-metric spaces can be obtained in an easier way. Our method is even easier than that of Du and Karapinar [11] and can be considered as a continuation of the papers [5, 10, 20].

2 Preliminaries

Throughout the paper, (E,t) will be a real Hausdorff topological vector space (abbr. *tvs*) with the zero vector denoted as θ. A *cone* in E is a proper nonempty and closed subset C which satisfies: 1° $C+C\subset C$, 2° $\lambda C\subset C$ for $\lambda\geq 0$, and 2° $C\cap(-C)=\{\theta\}$. If the cone C has a nonempty interior $\operatorname{int} C$ then it is called *solid*.

Example 1 *Some examples of solid cones are* $\{x=(x_i)_{i=1}^n\in\mathbb{R}^n : x_i\geq 0,\ i=1,\dots,n\}$ *in* $\mathbb{R}^n$ *and* $\{x\in C[a,b] : x(t)\geq 0,\ a\leq t\leq b\}$ *in* $C[a,b]$.

However, if one defines a cone in a similar manner in some other spaces, e.g., in c_0, l^p $(p>0)$, L^p $(p>0)$, *it appears to have empty interior [8, 31].*

Each cone C defines a partial order $\preceq$ on the vector space E by $x \preceq y \Leftrightarrow y - x \in C$. We will write $x \prec y$ when $x \preceq y$ and $x \neq y$. If the cone C is solid, $x \ll y$ denotes that $y - x \in \operatorname{int} C$. The triple (E,t,C) is an *ordered topological vector space*.

For a pair of elements $x, y \in E$ such that $x \preceq y$, put $[x,y] = \{z \in E : x \preceq z \preceq y\}$. The sets of the form $[x,y]$ are named order-intervals. A subset A of E is called order-convex if $[x,y] \subset A$ whenever $x,y \in A$ and $x \preceq y$. Ordered topological vector space is order-convex if it has a base of neighborhoods of θ consisting of order-convex subsets. In this case, the cone C is called *normal*. When E is a normed space, i.e., the topology t is induced by a norm $\|\cdot\|$, the last condition is fulfilled if and only if its unit ball is order-convex. This can be equivalently expressed as: there is a number m such that $x,y \in E$ and $\theta \preceq x \preceq y$ imply that $\|x\| \leq m\|y\|$. The minimal constant m satisfying the previous condition is called the *normal constant* of C. Obviously, the normal constant is always greater or equal to 1. In the case when $m = 1$, i.e., when $\theta \preceq x \preceq y$ implies that $\|x\| \leq \|y\|$, the cone C is called *monotone*.

Lemma 1 *([24], for the proof see [9, 31]) The following conditions are equivalent for a cone C in the normed space $(E, \|\cdot\|)$:*

(1) C is normal;

(2) there exists a norm $\|\cdot\|_1$ on E, equivalent to the given norm $\|\cdot\|$, such that the cone C is monotone w.r.t. $\|\cdot\|_1$.

The following example is classical.

Example 2 *Let $E = C^1_{\mathbb{R}}[0,1]$, with $\|x\| = \|x\|_\infty + \|x'\|_\infty$, $C = \{x \in E : x(t) \geq 0, t \in [0,1]\}$. This cone is solid but non-normal. Consider, for example, $x_n(t) = \frac{t^n}{n}$ and $y_n(t) = \frac{1}{n}$. Then $\theta \preceq x_n \preceq y_n$, and $\lim_{n\to\infty} y_n = \theta$, but $\|x_n\| = \max_{t\in[0,1]} \left|\frac{t^n}{n}\right| + \max_{t\in[0,1]} |t^{n-1}| = \frac{1}{n} + 1 > 1$; hence $\{x_n\}$ does not converge to zero. It follows that C is a non-normal cone.*

Definition 1 *[4, 10, 16, 21] Let X be a nonempty set and (E,t,C) be an ordered tvs. Suppose that a function $p : X \times X \to E$ satisfies the following conditions:*

(p1) $\theta \preceq p(x,y)$ for all $x,y \in X$ and $p(x,y) = \theta$ if and only if $x = y$;

(p2) $p(x,y) = p(y,x)$ for all $x,y \in X$;

(p3) $p(x,z) \preceq p(x,y) + p(y,z)$ for all $x,y,z \in X$.

Then p is called a tvs-cone metric and (X,p) is called a tvs-cone metric space.

3 Tvs-cone metric spaces

Let V be an absolutely convex and absorbing subset of a vector space E. Recall that its Minkowski functional is defined by $q_V(x) = \inf\{\lambda > 0 : x \in \lambda V\}$ for $x \in E$. It is a semi-norm on E (see, e.g. [30, II.1.5]) and $V \subset W$ implies that $q_W(x) \leq q_V(x)$ for $x \in E$.

Now, let (E,t,C) be an ordered *tvs* with C being solid, and let $e \in \operatorname{int} C$. Then $[-e,e] = (C-e) \cap (e-C) = \{z \in E : -e \preceq z \preceq e\}$ is an absolutely convex neighborhood of θ (for the proof see, e.g., [31, Proposition 2.2]). Its Minkowski functional $q_{[-e,e]}$ will be denoted by q_e. Note that $\operatorname{int}[-e,e] = (\operatorname{int} C - e) \cap (e - \operatorname{int} C)$, and q_e is an increasing function on C.

Lemma 2 *The functional q_e is a monotone norm on (E,C), i.e., C is a cone which is solid and normal w.r.t. q_e.*

Proof In order to prove that q_e is a norm on E, it suffices to show that $q_e(x) = 0$ implies that $x = \theta$. Suppose that $q_e(x) = 0$ for some $x \in E$. There is a sequence $\{\lambda_n\}$ of positive scalars such that $\lambda_n \to 0$ as $n \to \infty$ and $x \in \lambda_n[-e,e]$, i.e., $\lambda_n e - x \succeq \theta$ and $\lambda_n e + x \succeq \theta$. Passing to the limit as $n \to \infty$ and using that C is a cone in the given *tvs* (E,t), we get that $x = \theta$.

The obtained norm q_e is obviously monotone, i.e., $\theta \preceq x \preceq y$ implies that $q_e(x) \leq q_e(y)$.

Solidness and closedness of C w.r.t. q_e are proved, e.g., in [9, Proposition 19.9] and [1, Theorem 2.55].

□

Remark 1 *In the case when the given space E is normed, we obtain in this way two norms on it: the original norm and q_e. It is important to notice that if C is a normal cone w.r.t. the original norm, then these two norms are equivalent [9]. However, if this is not the case, then the two norms cannot be equivalent, because of Lemma 1 and Example 2.*

Note also that if e_1, e_2 are two points from $\operatorname{int} C$, then the norms q_{e_1} and q_{e_2} are always equivalent, i.e., they define the same topology. This topology is order-convex, i.e., it has a base of θ-neighborhoods consisting of order-convex subsets. This follows from the fact that if the cone C is normal (with the normal constant 1) and $e_1, e_2 \in \operatorname{int} C$, then each of the order-intervals $[-e_1,e_1]$, $[-e_2,e_2]$ absorbs the other.

Combining now Lemma 2 with Theorems 3.1 and 3.2 from [20], we deduce that:

(i) Each tvs-cone metric space with the underlying cone C that is solid in an ordered *tvs* (E,t,C) is in fact a cone metric space over (the same) cone C which is normal in an appropriate ordered normed space $(E, \|\cdot\|, C)$.

(ii) Each cone metric space over a solid cone is also a cone metric space over a solid and normal cone (in an appropriate norm).

We note that it is not clear whether these conclusions can be obtained using scalarization functions (which are not norms) that were utilized in several articles following [10].

We will apply now the obtained results to show how some cone metric versions of well known fixed point results can be obtained in an easy way. We first prove the following

Lemma 3 *If (E,t,C) is an ordered tvs with a solid cone C, and $e \in \operatorname{int} C$ is arbitrary, then*

$$\theta \preceq x \ll y \text{ implies that } q_e(x) < q_e(y).$$

Proof Suppose that $x, y \in E$ and $\theta \preceq x \ll y$. Then, considering the sequence $\{\frac{1}{n}y\}$ (obviously converging to θ) and $y - x \in \operatorname{int} C$, we get that

$$\frac{1}{n}y \ll y - x, \quad \text{i.e.,} \quad x \ll \left(1 - \frac{1}{n}\right)y,$$

for n sufficiently large. It follows that

$$q_e(x) \leq q_e\left(\left(1 - \frac{1}{n}\right)y\right) = \left(1 - \frac{1}{n}\right)q_e(y) < q_e(y).$$

□

Remark 2 *The previous lemma can be viewed as a refinement of [14, Lemma 2.1].*

In [16, Theorem 2], Huang and Zhang proved a cone metric version of celebrated Nemytzki-Edelstein fixed point theorem ([13, 27]) in the case when the underlying cone is regular. We will prove here that it holds true for arbitrary solid cones, with a modified contractive condition.

Theorem 1 *Let (X,p) be a tvs-cone metric space over a solid cone C in E, and let $f : X \to X$ be a mapping satisfying*

$$p(fx, fy) \ll p(x,y), \quad \text{for all } x \neq y. \tag{1}$$

Suppose that there is $x_0 \in X$ such that the respective Picard sequence $\{f^n x_0\}$ has a convergent subsequence (in particular, this is the case when (X,p) is sequentially compact). Then f has a unique fixed point, and for each positive integer n, it is $Fix(f) = Fix(f^n)$, i.e., f has the property P.

Proof Take arbitrary $e \in \operatorname{int} C$ and form the respective monotone norm q_e on E according to Lemma 2. Denote $d = q_e \circ p$, i.e., $d(x,y) = q_e(p(x,y))$ for $x, y \in X$. Then d is a metric on X (by [20, Theorem 3.1]) and, using Lemma 3, the relation (1) implies that

$$d(fx, fy) < d(x,y), \quad \text{for all } x \neq y.$$

Now the proof proceeds similarly as in the metric case.

We will prove just the property P. Suppose that $u \in Fix(f^n)$, but $u \notin Fix(f)$, where n is the smallest such index. Then

$$d(u, fu) = d(f^n u, f^{n+1} u) < d(f^{n-1} u, f^n u) < \cdots < d(u, fu),$$

a contradiction. Hence, $u \in Fix(f)$, i.e., $Fix(f^n) \subset Fix(f)$. The reverse inclusion is obvious.

□

Formulated in another way:

Corollary 1 *The scalar and vector (with $\ll$ used in the condition* (1)*) versions of Nemytzki-Edelstein fixed point theorem are equivalent.*

Question 1 *Do the conclusions of Theorem 1 hold if the cone C is just solid (and not regular) and $\ll$ in* (1) *is replaced by $\prec$?*

4 Tvs-cone b-metric spaces

b-metric spaces (sometimes called metric-type spaces), as another generalization of metric spaces were first considered by I.A. Bakhtin [3] and Czerwik [8]. Cone b-metric spaces were introduced in [7] and [18]. Extension to *tvs* case can be done in an obvious way:

Definition 2 *Let X be a nonempty set, (E, t, C) be an ordered tvs and $s \geq 1$ be a given real number. A function $p_s : X \times X \to E$ is called a tvs-cone b-metric and (X, p_s) is called a tvs-cone b-metric space if the following conditions hold:*

(ps1) $\theta \preceq p_s(x, y)$ *for all* $x, y \in X$ *and* $p_s(x, y) = \theta$ *if and only if* $x = y$;

(ps2) $p_s(x, y) = p_s(y, x)$ *for all* $x, y \in X$;

(ps3) $p_s(x, z) \preceq s[p_s(x, y) + p_s(y, z)]$ *for all* $x, y, z \in X$.

If $(E, \|\cdot\|, C)$ is an ordered normed space, then p_s is called a cone b-metric, and (X, d) is a cone b-metric space.

Obviously, for $s = 1$, (tvs)-cone b-metric space is a (tvs)-cone metric space.

Most of the standard notions concerning convergence of sequences can be introduced in these spaces in the usual way. The main obstacle in deriving results is the fact that a (cone) b-metric is not always a continuous function (in the sense that $x_n \to x$ and $y_n \to y$ imply that $p_s(x_n, y_n) \to p_s(x, y)$ as $n \to \infty$), see, e.g., [17, Example 2].

Lemma 4 *Let (X, p_s) be a tvs-cone b-metric space (with the parameter s) over a solid cone C, $e \in intC$ and let q_e be the corresponding Minkowski functional of $[-e, e]$. Then $d_q = q_e \circ p_s$ is a b-metric on X (with the same parameter s).*

Proof Clearly, $d_q(x,y) = d_q(y,x)$ for all $x,y \in X$ and $x = y$ implies that $d_q(x,y) = 0$. Also, since q_e is a semi-norm and p_s is a tvs-cone b-metric, we have

$$q_e(p_s(x,z)) \leq s(q_e(p_s(x,y)) + q_e(p_s(y,z))),$$

i.e.,

$$d_q(x,z) \leq s[d_q(x,y) + d_q(y,z)]$$

for all $x,y,z \in X$. Now, we prove that $d_q(x,y) = 0$ implies that $x = y$. Let $q_e \circ p_s(x,y) = 0$. Then $\inf\{\lambda > 0 : p_s(x,y) \in \lambda[-e,e]\} = 0$. Thus, there exists a sequence of positive scalars $\lambda_n \to 0$ such that $p_s(x,y) \in \lambda_n[-e,e]$. Suppose, to the contrary, that $x \neq y$. Then, since $\theta \prec p_s(x,y) \preceq \lambda_n e$, for each $c \in \mathrm{int}C$ there exists n_0 such that $p_s(x,y) \ll c$ for $n \geq n_0$. Since c is an arbitrary interior point of the cone C it follows that $p_s(x,y) = \theta$. This is a contradiction. Thus, the proof of lemma is complete.

□

Remark 3 *Under the assumptions from the previous lemma, similarly as in [20, Theorem 3.2], the following properties can be easily deduced:*

(i) A sequence $\{x_n\}$ converges to x in (X, p_s) if and only if $d_q(x_n, x) \to 0$ as $n \to \infty$;

(ii) $\{x_n\}$ is a Cauchy sequence in (X, p_s) if and only if $\{x_n\}$ is a Cauchy sequence in (X, d_q);

(iii) (X, p_s) is complete if and only if (X, d_q) is complete.

As a sample, we show how a tvs-cone b-metric version of Banach Contraction Principle can be easily deduced.

Theorem 2 *Let (X, p_s) be a complete tvs-cone b-metric space with $s \geq 1$ and $\lambda \in [0,1)$. If $f : X \to X$ satisfies the contractive condition*

$$p_s(fx, fy) \preceq \lambda p_s(x,y), \tag{2}$$

for all $x,y \in X$, then f has a unique fixed point in X. Moreover, for each $x \in X$, the Picard sequence $\{f^n x\}$ converges to the unique fixed point of f.

Proof Set $d_q = q_e \circ p_s$. Lemma 4 and Remark 3 imply that (X, d_q) is a complete b-metric space. Also, we conclude that (2) implies that

$$d_q(fx, fy) \leq \lambda d_q(x, y)$$

for all $x, y \in X$. Thus, the conclusion follows from [12, Theorem 2.1].

□

In a similar way, a lot of fixed point results under various contractive conditions can be proved in tvs-cone b-metric spaces, by reducing them to the respective results in b-metric spaces. Note that most of them need a modification compared with the respective standard metric results, in the sense that conditions on contractive constants depend on parameter s (the case of Banach principle is an exception), see the respective discussion in [12].

We finish by applying our previous results in order to prove tvs-cone b-metric version of Nemytzki-Edelstein theorem in the case when tvs-cone b-metric is continuous. For discussion about (sequential) compactness in these spaces see [18].

Theorem 1 *Let (X, p_s) be a tvs-cone b-metric space over a solid cone C in E, such that the tvs-cone b-metric p_s is continuous. Let $f : X \to X$ be a mapping satisfying*

$$p_s(fx, fy) \ll p_s(x, y), \quad \text{for all } x \neq y. \tag{3}$$

Suppose that there is $x_0 \in X$ such that the respective Picard sequence $\{f^n x_0\}$ has a convergent subsequence (in particular, this is the case when (X, p_s) is sequentially compact). Then f has a unique fixed point.

Proof Similarly as in the proof of Theorem 1, take arbitrary $e \in \operatorname{int} C$ and form the respective monotone norm q_e on E according to Lemma 2. Denote $d_q = q_e \circ p_s$, i.e., $d_q(x, y) = q_e(p_s(x, y))$ for $x, y \in X$. Then, by Lemma 4, d_q is a b-metric on X (with the same parameter s as p_s). Moreover, d_q is continuous, together with p_s. Applying Lemma 3, the relation (3) implies that

$$d_q(fx, fy) < d_q(x, y), \quad \text{for all } x \neq y.$$

Now the proof proceeds similarly as in the b-metric case (see, e.g., [6, Theorem 3.1]).

□

Question 2 *Does the previous result hold true if the b-metric p_s is not continuous?*

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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