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α_β-Contractive Mappings and Upclass of Type II Function

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Abstract: Inspired by the recent result of N. Hussain et al. [N. Hussain, E. Karapinar, P. Salimi, F. Akbar, $\alpha-$admissible mappings and related fixed point theorems, Journal of Ineaqualities and Applications 2013, 2013:114] we introduced two new type functions, which we denoted as upclass of type *II* and α_β-contractive mappings. Specifically, all results of the mentioned paper is possible to generalize using this two types of functions.

Keywords: Complete metric space; altering distance function; α_β-contractive mapping; fixed point.

1 Introduction and Preliminaries

Fixed point theory is one of the most significant branches of nonlinear analysis. Banach contraction principle [2], remarkable for simplicity is one of the most quoted in the overall results of the analysis with a special application of the theory of differential and integral equations. In the proof of this famous theorem he presented the existence and uniqueness of a fixed point, and also an iterative procedure for the construction of a fixed point. There are a lot of papers on the topic of generalization of Banach contraction principle. In attempt to generalize the mentioned principle, many researchers generalize the following result. One of the generalization is presented by the following fixed point theorem:

Theorem 1.1 *(see, [1, 3, 4, 6]) Let (X,d) be a complete metric space and $T : X \to X$ be a mapping. Assume that there exists a function $\beta : [0,\infty) \to [0,1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$ and $d(Tx,Ty) \leq \beta(d(x,y))d(x,y)$ for all $x, y \in X$. Then T has a unique fixed point.*

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N. Hussein at all. [5] generalized above mentioned result using $\alpha-$admissible mappings.

Definition 1.2 *[7] Let $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}^+$. We say that T is an $\alpha-$admissible mapping if*

$$\alpha(x,y) \geq 1 \textit{ implies } \alpha(Tx,Ty) \geq 1 \ x,y \in X.$$

Their results are the following fixed point theorems.

Theorem 1.3 *Let (X,d) be a complete metric space and $T : X \to X$ be an $\alpha-$admissible mapping. Assume that there exists a function $\beta : [0,\infty) \to [0,1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$ and*

$$(d(Tx,Ty)+l)^{\alpha(x,Tx)\alpha(y,Ty)} \leq \beta(d(x,y))d(x,y)+l \tag{1}$$

for all $x,y \in X$ where $l > 1$. Suppose that either

(a) T is continuous, or

(b) if $\{x_n\}$ is a sequence in X such that $x_n \to x$, $\alpha(x_n,x_{n+1}) \geq 1$ for all n, then $\alpha(x,fx) \geq 1$.

If there exists $x_0 \in X$ such that $\alpha(x_0,fx_0) \geq 1$, then T has a fixed point.

Remark 1.4 *It is obvious that in the case when $\alpha(x,Tx)\alpha(y,Ty) = 1$ Theorem 1.1 and Theorem 1.3 are the same.*

Theorem 1.5 *Let (X,d) be a complete metric space and $T : X \to X$ be an $\alpha-$admissible mapping. Assume that there exists a function $\beta : [0,\infty) \to [0,1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$ and*

$$(\alpha(x,Tx)\alpha(y,Ty)+1)^{d(Tx,Ty)} \leq 2^{\beta(d(x,y))d(x,y)} \tag{2}$$

for all $x,y \in X$ where $l > 1$. Suppose that either

(a) T is continuous, or

(b) if $\{x_n\}$ is a sequence in X such that $x_n \to x$, $\alpha(x_n,x_{n+1}) \geq 1$ for all n, then $\alpha(x,fx) \geq 1$.

If there exists $x_0 \in X$ such that $\alpha(x_0,fx_0) \geq 1$, then T has a fixed point.

Remark 1.6 *It is obvious that in the case when $\alpha(x,Tx)\alpha(y,Ty) = 1$ Theorem 1.5 is reduced to Theorem 1.1.*

Theorem 1.7 *Let (X,d) be a complete metric space and $T:X\to X$ be an $\alpha-$admissible mapping. Assume that there exists a function $\beta:[0,\infty)\to[0,1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n)\to 1$ implies $t_n\to 0$ and*

$$\alpha(x,Tx)\alpha(y,Ty)d(Tx,Ty)\le\beta(d(x,y))d(x,y) \tag{3}$$

for all $x,y\in X$ where $l>1$. Suppose that either

(a) T is continuous, or

(b) if $\{x_n\}$ is a sequence in X such that $x_n\to x$, $\alpha(x_n,x_{n+1})\ge 1$ for all n, then $\alpha(x,fx)\ge$ 1.

If there exists $x_0\in X$ such that $\alpha(x_0,fx_0)\ge 1$, then T has a fixed point.

Remark 1.8 *It is obvious that in the case when $\alpha(x,Tx)\alpha(y,Ty)=1$ Theorem 1.5 is reduced to Theorem 1.1.*

2 Main results

In this paper, in order to generalized conditions (1) and (2) we introduced the concepts of pair (f,h) which we noted as upclass of type *II*, as well as the notion of α_β-contractive mappings.

First of all we define a function of subclass of type *II* in the following way:

Definition 2.1 *We say that $h:\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ is a function of subclass of type II if is continuous and*

$$x,y\ge 1\Longrightarrow h(1,1,z)\le h(x,y,z)$$

Example 2.2 *Let $x,y,z\in R^+$. Then*

(1) $h(x,y,z)=(z+l)^{xy}$, $l>1$,

(2) $h(x,y,z)=(xy+l)^z$, $l>0$,

(3) $h(x,y,z)=z$,

(4) $h(x,y,z)=x^my^nz^p$, $m,n,p\in\mathbb{N}$,

(5) $h(x,y,z)=\frac{x^m+x^ny^p+y^q}{3}z^k$, $m,n,p,q,k\in\mathbb{N}$.

Definition 2.3 *Let $f:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$. We say that the pair (f,h) is upclass of type II if f is a continuous function, h is a subclass of type II and*

$$0\le s\le 1\Rightarrow f(s,t)\le f(1,t),$$

$$h(1,1,z) \leq f(s,t) \Rightarrow z \leq st.$$

Example 2.4 *Let $s \in [0,1]$ and $t \in R^+$. Then*

(1) $h(x,y,z) = (z+l)^{xy}$, $l > 1$, $f(s,t) = st + \frac{l}{e}, e \geq 1$,

(2) $h(x,y,z) = (xy+l)^z$, $l > 0$, $f(s,t) = (1+l)^{st}$,

(3) $h(x,y,z) = z$, $f(s,t) = st$,

(4) $h(x,y,z) = x^m y^n z^p$, $m,n,p \in \mathbb{N}$, $f(s,t) = s^p t^p$,

(5) $h(x,y,z) = \frac{x^m + x^n y^p + y^q}{3} z^k$, $m,n,p,q,k \in \mathbb{N}$, $f(s,t) = s^k t^k$.

Definition 2.5 *A function $\psi : [0,\infty) \to [0,\infty)$ is called altering distance function if the following properties are satisfied:*

1. *ψ is continuous and non-decreasing;*
2. $\psi^{-1}(\{0\}) = 0$.

We denote by Ψ the set of all altering distance functions.

Definition 2.6 *Let (X,d) be a metric space, $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}^+$. A mapping T is said to be α_β-contractive mapping if there exists a $\beta : [0,\infty) \to [0, 1)$ with the property that $t_n \to 0$ whenever $\beta(t_n) \to 1$ as well as for all $x,y \in F$, following condition holds:*

$$h(\alpha(x,Tx), \alpha(y,Ty), \psi(d(Tx,Ty))) \leq f(\beta(d(x,y)), \psi(d(x,y))), \tag{4}$$

where pair (f,h) is a upclass of type II and $\psi \in \Psi$.

Remark 2.7 *If $h(x,y,z) = (z+l)^{xy}$, $l > 1$, $f(x,y) = xy + l$ and $\psi(t) = t$, in condition (4), we have condition (1).*

Remark 2.8 *If $h(x,y,z) = (xy+1)^z$, $f(x,y) = (1+m)^{xy}$, $m=1$ and $\psi(t) = t$, in (4), we have condition (2).*

Remark 2.9 *If $h(x,y,z) = xyz$, $f(x,y) = xy$ and $\psi(t) = t$, for all $x,y,z \in X$ and $t > 0$ in (4), we have condition (3).*

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