SCIENTIFIC PUBLICATIONS OF THE STATE UNIVERSITY OF NOVI PAZAR
SER. A: APPL. MATH. INFORM. AND MECH. vol. 9, 1 (2017), 19-24.

Remark on Lower Bound for Forgotten Topological Index

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Abstract: Let G be a simple connected graph with n vertices and m edges with vertex degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n > 0$. Denote by $F = \sum_{i=1}^{n} d_i^3$ forgotten topological index of graph G. In this paper we give some lower bounds for invariant F. Also, obtained bounds are compared with some known bounds from the literature.

Keywords: Vertex degree, the first Zagreb index, forgotten topological index

1 Introduction

Let G be a simple connected graph with n vertices and m edges. Denote by $d_1 \geq d_2 \geq \cdots \geq d_n > 0$ a sequence of vertex degrees of graph G. Throughout this paper we use standard notation: $\Delta = d_1$, $\Delta_2 = d_2$, and $\delta = d_n$.

In [5] vertex-degree-based topological indices, named the first and the second Zagreb indices M_1 and M_2, were defined as

$$M_1 = M_1(G) = \sum_{i=1}^{n} d_i^2 \quad \text{and} \quad M_2 = M_2(G) = \sum_{i \sim j} d_i d_j,$$

where $i \sim j$ denotes the adjacency of the vertices i and j in graph G.

Details on these topological indices can be found in [1, 2, 6, 7].

In [4] (see also [6]) forgotten topological index F was defined as

$$F = F(G) = \sum_{i=1}^{n} d_i^3.$$

Let $E = \{e_1, e_2, \dots, e_m\}$ be a set of edges of graph G and $d(e_1) \geq d(e_2) \geq \cdots \geq d(e_m)$ sequence of edge degrees. In [10], an edge-degree graph topological index, named refor-

Manuscript received September 12, 2016; accepted January 23, 2017.

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mulated Zagreb index, EM_1, is defined as

$$EM_1 = EM_1(G) = \sum_{i=1}^{m} d(e_i)^2.$$

Of course, it is easy to note that EM_1 is not new topological index, since it is the first Zagreb index for a line-graph $L = L(G)$ of graph G.

In this paper we state two inequalities that set lower bounds for invariant F in terms of topological index M_1 and graph parameters m, Δ, Δ_2, and δ. Obtained results will be used to determine lower bounds for topological indices EM_1 and M_2.

2 Preliminaries

In this section we give some known results for invariants F, M_1 and EM_1 that will be needed in the subsequent considerations.

In [4] the following inequality for graph invariant F was proved

$$F \geq \frac{M_1^2}{2m}, \tag{1}$$

with equality if and only if G is a regular graph.

The following equality was proved in [15] for graph invariant EM_1

$$EM_1 = F + 2M_2 - 4M_1 + 4m. \tag{2}$$

In [11] it was proved

$$EM_1 \geq \frac{M_1^2}{2m} + 2M_2 - 4M_1 + 4m. \tag{3}$$

Equality holds if and only if $L(G)$ is regular.

In [9] it was proved

$$F \leq \frac{\Delta^2 + \delta^2}{\Delta\delta} M_2. \tag{4}$$

3 Main result

The following theorem establishes lower bound for invariant F in terms of topological index M_1 and graph parameters m, Δ and Δ_2.

Theorem 3.1. *Let G be a simple connected graph with n, $n \geq 2$, vertices and m edges. Then*

$$F \geq \frac{M_1^2}{2m} + \frac{\Delta\Delta_2 (\Delta - \Delta_2)^2}{2m}. \tag{5}$$

Equality holds if and only if G is regular graph.

Proof. Let $p = (p_i)$, $i = 1, 2, \ldots, m$, be positive real number sequence, and $a = (a_i)$ and $b = (b_i)$, $i = 1, 2, \ldots, m$, sequences of non-negative real numbers of similar monotonicity. In [14] (see also [13]) it was proved that

$$T_n(a,b;p) \geq T_{n-1}(a,b;p), \quad n \geq 2, \tag{6}$$

where

$$T_n(a,b;p) = \sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i.$$

From (6) it follows

$$T_n(a,b;p) \geq T_{n-1}(a,b;p) \geq \cdots \geq T_2(a,b;p) \geq 0.$$

Since

$$\begin{aligned} T_2(a,b;p) &= \sum_{i=1}^{2} p_i \sum_{i=1}^{2} p_i a_i b_i - \sum_{i=1}^{2} p_i a_i \sum_{i=1}^{2} p_i b_i \\ &= (p_1 + p_2)(p_1 a_1 b_1 + p_2 a_2 b_2) - (p_1 a_1 + p_2 a_2)(p_1 b_1 + p_2 b_2) \\ &= p_1 p_2 (a_1 - a_2)(b_1 - b_2), \end{aligned}$$

we have that

$$\sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i a_i b_i \geq \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i + p_1 p_2 (a_1 - a_2)(b_1 - b_2). \tag{7}$$

For $p_i = a_i = b_i = d_i$, $i = 1, 2, \ldots, n$, this inequality becomes

$$\sum_{i=1}^{n} d_i \sum_{i=1}^{n} d_i^3 \geq \left(\sum_{i=1}^{n} d_i^2 \right)^2 + d_1 d_2 (d_1 - d_2)^2,$$

wherefrom we get (5). □

Remark 3.2. *Since*

$$\frac{M_1^2}{2m} + \frac{\Delta\Delta_2 \left(\Delta - \Delta_2\right)^2}{2m} \geq \frac{M_1^2}{2m},$$

the inequality (5) *is stronger than* (1).

Corollary 3.3. *Let G be a simple connected graph with n, $n \geq 2$, vertices and m edges. Then*

$$F \geq \frac{8m^3}{n^2} + \frac{\Delta\Delta_2 \left(\Delta - \Delta_2\right)^2}{2m}, \tag{8}$$

with equality if and only if G is regular graph.

Proof. Inequality (8) is a direct consequence of (5) and the following inequality

$$M_1 \geq \frac{4m^2}{n}, \tag{9}$$

proved in [3]. □

Corollary 3.4. *Let G be a simple connected graph with n, $n \geq 2$, vertices and m edges. Then*

$$M_2 \geq \frac{\Delta\delta}{2m(\Delta^2+\delta^2)}\left(M_1^2+\Delta\Delta_2(\Delta-\Delta_2)^2\right), \tag{10}$$

with equality if and only if G is regular graph.

Corollary 3.5. *Let G be a simple connected graph with n, $n \geq 2$, vertices and m edges. Then*

$$EM_1 \geq \frac{M_1^2}{2m}+2M_2-4M_1+4m+\frac{\Delta\Delta_2(\Delta-\Delta_2)^2}{2m}, \tag{11}$$

with equality if and only if G is regular.

Remark 3.6. *Since*

$$\frac{\Delta\Delta_2(\Delta-\Delta_2)^2}{2m} \geq 0,$$

the inequality (11) *is stronger than* (3).

Remark 3.7. *Note that inequality* (7) *is a generalization of Chebyshev inequality (see for example [12]).*

Theorem 3.8. *Let G be a simple connected graph with n, $n \geq 3$, vertices and m edges. Then*

$$F \geq \delta^3+\frac{(M_1-\delta^2)^2}{2m-\delta}+\frac{\Delta\Delta_2(\Delta-\Delta_2)^2}{2m-\delta}. \tag{12}$$

Equality holds if and only if G is regular graph.

Proof. According to (7) we have that

$$\sum_{i=1}^{n-1} p_i \sum_{i=1}^{n-1} p_i a_i b_i \geq \sum_{i=1}^{n-1} p_i a_i \sum_{i=1}^{n-1} p_i b_i + p_1 p_2 (a_1-a_2)(b_1-b_2).$$

Putting $p_i = a_i = b_i = d_i$, $i = 1, 2, \ldots, n-1$, in this inequality, we get

$$\sum_{i=1}^{n-1} d_i \sum_{i=1}^{n-1} d_i^3 \geq \left(\sum_{i=1}^{n-1} d_i^2\right)^2 + d_1 d_2 (d_1-d_2)^2,$$

i.e.

$$(2m-\delta)(F-\delta^3) \geq (M_1-\delta^2)^2+\Delta\Delta_2\,(\Delta-\Delta_2)^2,$$

wherefrom we obtain inequality (12). □

Corollary 3.9. *Let G be a simple connected graph with n, $n \geq 3$, vertices and m edges. Then*

$$F \geq 2m\delta^2 + \frac{\Delta\Delta_2(\Delta-\Delta_2)^2}{2m-\delta},$$

with equality if and only if G is regular.

Corollary 3.10. *Let G be a simple connected graph with n, $n \geq 3$, vertices and m edges. Then*

$$M_2 \geq \frac{\Delta\delta}{\Delta^2+\delta^2}\left(2m\delta^2 + \frac{\Delta\Delta_2(\Delta-\Delta_2)^2}{2m-\delta}\right).$$

Corollary 3.11. *Let G be a simple connected graph with n, $n \geq 3$, vertices and m edges. Then*

$$EM_1 \geq \delta^3 + \frac{(M_1-\delta^2)^2}{2m-\delta} + \frac{\Delta\Delta_2(\Delta-\Delta_2)^2}{2m-\delta} + 2M_2 - 4M_1 + 4m,$$

with equality if and only if G is regular.

Theorem 3.12. *Let G be a simple connected graph with n, $n \geq 2$, vertices and m edges. Then*

$$M_1 \geq \frac{4m^2}{n} + \frac{(\Delta-\Delta_2)^2}{n}. \tag{13}$$

Equality holds if and only if G is regular graph.

Proof. For $p_i = 1$, $a_i = b_i = d_i$, $i = 1, 2, \ldots, n$, inequality (7) becomes

$$n\sum_{i=1}^{n} d_i^2 \geq \left(\sum_{i=1}^{n} d_i\right)^2 + (\Delta-\Delta_2)^2,$$

i.e.

$$nM_1 \geq 4m^2 + (\Delta-\Delta_2)^2,$$

wherefrom we obtain (13). □

Remark 3.13. *Since* $(\Delta-\Delta_2)^2 \geq 0$*, the inequality* (13) *is stronger than* (9).

By a similar procedure as in case of Theorem 3.12, the following statement can be proved.

Theorem 3.14. *Let G be a simple connected graph with n, $n \geq 3$, vertices and m edges. Then*

$$M_1 \geq \delta^2 + \frac{(2m-\delta)^2 + (\Delta-\Delta_2)^2}{n-1}.$$

Equality holds if and only if G is regular.

References

[1] B. Borovićanin, K. C. Das, B. Furtula, I. Gutman, *Zagreb indices: Bounds and Extremal graphs, In: Bounds in Chemical Graph Theory – Basics, (I. Gutman, B. Furtula, K. C. Das, E. Milovanović, I. Milovanović, Eds.)*, Mathematical Chemistry Monographs, MCM 19, Univ. Kragujevac, Kragujevac, 2017, pp. 67–153.

[2] B. Borovićanin, K. C. Das, B. Furtula, I. Gutman, *Bounds for Zagreb indices*, MATCH Commun. Math. Comput. Chem., 78 (2017) 17–100.

[3] C. S. Edwards, *The largest vertex degree sum for a triangle in a graph*, Bull. London Math. Soc., 9 (1977) 203–208.

[4] B. Furtula, I. Gutman, *A forgotten topological index*, J. Math. Chem., 53 (2015) 1184–1190.

[5] I. Gutman, N. Trinajstić, *Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons*, Chem. Phys. Lett., 17 (1972), 535–538.

[6] I. Gutman, *On the origin of two degree-based topological indices*, Bull. Acad. Serbie Sci. Arts (Ch. Sci. Math. Natur.), 146 (2014) 39–52.

[7] I. Gutman, K. C. Das, *The first Zagreb index 30 years after*, MATCH Commun. Math. Comput. Chem., 50 (2004) 83–92.

[8] I. Gutman, B. Furtula, Ž. Kovijanić Vukićević, G. Popivoda, *On Zagreb indices and coindices*, MATCH Commun. Math. Comput. Chem., 74 (2015) 5–16.

[9] B. Liu, I. Gutman, *On general Randić indices*, MATCH Commun. Math. Comput. Chem., 58 (2007) 147–154.

[10] A. Milićević, S. Nikolić, N. Trinajstić, *On reformulated Zagreb indices*, Mol. Divers., 8 (2004) 393–399.

[11] E. I. Milovanović, I. Ž. Milovanović, E. Ć. Dolićanin, E. Glogić, *A note on the first reformulated Zagreb index*, Appl. Math. Comput., 273 (2016) 16–20.

[12] D. S. Mitrinović, P. M. Vasić, *Analytic inequalities*, Springer Verlag, Berlin-Heidelberg-New York, 1970.

[13] D. S. Mitrinović, P. M. Vasić, *History, variations and generalisations of the Čebyšev inequality and the question of some priorities*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz., 461–497 (1974), 1–30.

[14] P. M. Vasić, R. Ž. Djordjević, *Čebyšev inequality for convex sets*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz., 412–460 (1973), 17–20.

[15] B. Zhou, N. Trinajstić, *Some properties of the reformulated Zagreb indices*, J. Math. Chem., 48 (2010), 714–719.