SCIENTIFIC PUBLICATIONS OF THE STATE UNIVERSITY OF NOVI PAZAR
SER. A: APPL. MATH. INFORM. AND MECH. vol. 9, 1 (2017), 47-57.

Finite Fourier Decomposition of Signals Using Generalized Difference Operator

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Abstract: In this paper, we introduce discrete inner product of two functions, discrete orthogonal and orthonormal system of functions and develop finite Fourier series for polynomial factorial, polynomial, exponential, rational and logarithm functions using the inverse of generalized difference operator

Keywords: Discrete inner product, Discrete orthonormal system, Finite Fourier series and Generalized difference operator.

1 Introduction

In Fourier analysis, a signal is decomposed into its constituent sinusoids. In the reverse by operating the inverse Fourier transform, the signal can be synthesized by adding up its constituent frequencies. Many signals that we encounter in daily life such as speech, automobile noise, chirps of birds, music etc. have a periodic or quasi-periodic structure, and that the cochlea in the human hearing system performs a kind of harmonic analysis of the input audio signals in biological and physical systems [11]. The Fourier series decomposes the given input signals into a sum of sinusoids. By removing the high frequency terms(noise) of Fourier series and then adding the remaining terms can yield better signals [4].

Finite Fourier series is a powerful tool for attacking many problems in the theory of numbers. It is related to certain types of exponential and trigonometric sums. It may therefore be expanded into a finite Fourier series of the form
$f(\alpha^{\mu}) = \sum_{j=0}^{m-1} g(j)\alpha^{\mu j}(\mu = 0, 1, \cdots, m-1)$. The orthogonality relation
$\sum_{j=0}^{m-1} \alpha^{aj}\alpha^{-bj} = \begin{cases} m & (a \equiv b(mod\ m)), \\ 0 & (a \not\equiv b(mod\ m)), \end{cases}$ enables us to determine the finite Fourier coef-

Manuscript received March 21, 2016; accepted August 29, 2016.
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ficients $g(k)$ explicitly by means of the formula $g(k) = \frac{1}{m}\sum_{\mu=0}^{m-1} f(\alpha^{\mu})\alpha^{-\mu k}$ [2]. If we are given k distinct complex numbers $z_0, z_1, \cdots, z_{k-1}$, then there is one and only one polynomial $P(x) = \zeta_0 + \zeta_1 x + \cdots + \zeta_{k-1}x^{k-1}$ satisfying the equations $P(\omega_\nu) = z_\nu(\nu = 0, 1, \cdots, k-1)$ [10].

A finite Fourier series: $\eta(t) = A_0 + \sum_{q=1}^{N/2} A_q \cos(q\sigma_1 t) + \sum_{q=1}^{N/2-1} B_q \sin(q\sigma_1 t)$, where η = sea surface elevation (m), t = time (s), A_0 = recond mean (m), N = total number of sampling points, A_q and B_q = Fourier coefficients (m), q = harmonic component index (in the frequency domain) and σ_1 = fundamental radian frequency, is used in [6]. The sum of N sine waves defined over the time interval, $0 \le t \le T : y = \sum_{n=1}^{N} a_n \cos(\omega_n t + \phi_n), 0 \le t_n \le T$, $a_n \ge 0$, $0 \le \phi_n < 2\pi$, where a_n is amplitude and t is time, is also a finite Fourier series[3]. In [7], the authors describe an efficient formulation, based on a discrete Fourier series expansion, of the analysis of axi-symmetric solids subjected to non-symmetric loading. They have discussed the Fourier series approach, the discrete Fourier series representation problems such as the presence of Gibb's phenomenon and the lack of conformity of elements. Here we arrive a new type of finite Fourier series for functions (signals) by defining discrete orthonormal family of functions using inverse generalized difference operator Δ_ℓ^{-1}. This finite Fourier series becomes Fourier series as ℓ tends to zero. Suitable examples verified by MATLAB are inserted to illustrate the findings.

2 Preliminaries

The primitive N^{th} roots of unity $(z^N = 1 \; but \; z^r \neq 1; 0 < r < N)$

$$z_n = e^{i(2\pi/N)n}, \; n = 1, 2, 3, ..., N-1, \tag{1}$$

where n and N are co-prime, satisfies the geometric series expressed as

$$\sum_{k=0}^{N-1} z_n^k = \Delta^{-1} z_n^k \Big|_{k=0}^{N} = \frac{z_n^N - 1}{z_n - 1} = \begin{cases} 1 & if \quad N = 1 \\ 0 & if \quad N > 1. \end{cases} \tag{2}$$

From (1) and (2), the complex discrete-time sequence $e_r(k)$ is defined as

$$e_n(k) = (z_n)^k = e^{i(2\pi/N)nk}; \; n, \; k = 0, 1, 2, ..., N-1. \tag{3}$$

For the positive integers n, r and N, the $e_n(k)$ defined in (3) satisfies the identity

$$\sum_{k=0}^{N-1} e_n(k) = \Delta^{-1} e_n(k) \Big|_{k=0}^{N} = \Delta^{-1} e^{i(2\pi n/N)k} \Big|_{k=0}^{N} = \begin{cases} N & if \quad n = rN \\ 0 & if \quad n \neq rN. \end{cases} \tag{4}$$

This mathematical property is utilized with the factorization into two orthogonal exponential functions, $\{e_n(k)\}$ satisfying

$$\Delta^{-1} e_n(k) e_m^*(k)\Big|_{k=0}^{N} = \Delta^{-1} e^{i(\frac{2\pi(n-m)k}{N})}\Big|_{k=0}^{N} = \begin{cases} N & if \quad n-m = rN \\ 0 & if \quad n-m \neq rN, \end{cases} \tag{5}$$

where m,n and r are integers, and the notation $(*)$ represents the complex conjugate. The equation (5) induces us to define a generalized discrete orthonormal system and a finite Fourier series by replacing Δ by Δ_ℓ and $e_n(k)$ by $u_n(k)$. Nonexistence of solutions of certain type of second order generalized α-difference equation with the operator $\Delta_{\alpha(\ell)}$ has been discussed in [9]. When $\alpha = 1$ the operator $\Delta_{\alpha(\ell)}$ becomes the generalized difference operator Δ_ℓ.

Definition 1 *[8] Let $u(k),\ k \in [0,\infty)$, be a real or complex valued function and $\ell > 0$ be fixed. Then, the generalized difference operator Δ_ℓ on $u(k)$ is defined as*

$$\Delta_\ell u(k) = u(k+\ell) - u(k), \tag{6}$$

and its inverse is defined as if there is a function $v(k)$ such that

$$\Delta_\ell v(k) = u(k),\ then\ v(k) \ = \ \Delta_\ell^{-1} u(k) + c_j, for\ k \in \Big\{ j + r\ell \Big\}_{r=0}^{\infty}, \tag{7}$$

where c_j is constant, $j = k - [k/\ell]\ell$ and $[k/\ell]$ is the integer part of k/ℓ.

Lemma 1 *[8] Let s_r^m and S_r^m are the Stirling numbers of first and second kinds, $k_\ell^{(0)} = 1$, $k_\ell^{(1)} = k$ and $k_\ell^{(m)} = k(k-\ell)(k-2\ell)\cdots(k-(m-1)\ell)$. Then we have*

$$k_\ell^{(m)} = \sum_{r=1}^{m} s_r^m \ell^{m-r} k^r,\ k^m = \sum_{r=1}^{m} S_r^m \ell^{m-r} k_\ell^{(r)},\ \Delta_\ell k_\ell^{(m)} = (m\ell) k_\ell^{(m-1)} \tag{8}$$

and

$$\Delta_\ell^{-1} k_\ell^{(m)} = \frac{k_\ell^{(m+1)}}{\ell(m+1)},\ \Delta_\ell^{-1} k^m = \sum_{r=1}^{m} \frac{S_r^m \ell^{m-r} k_\ell^{(r)}}{(r+1)\ell}. \tag{9}$$

Lemma 2 *[5] Let p be real, $\ell > 0$, $k \in (\ell,\ \infty)$ and $p\ell \neq m2\pi$. Then, we have*

$$\Delta_\ell^{-1} \sin pk = \frac{\sin p(k-\ell) - \sin pk}{2(1-\cos p\ell)} + c_j, \tag{10}$$

$$\Delta_\ell^{-1} \cos pk = \frac{\cos p(k-\ell) - \cos pk}{2(1-\cos p\ell)} + c_j \tag{11}$$

and

$$\Delta_\ell^{-1}(u(k)w(k)) = v(k)\Delta_\ell^{-1} w(k) - \Delta_\ell^{-1}(\Delta_\ell^{-1} w(k+\ell)\Delta_\ell u(k)). \tag{12}$$

Remark 1 *From (10) and (11), we have*

$$\Delta_\ell^{-1}\sin pk\Big|_0^{2\pi} = 0 = \Delta_\ell^{-1}\cos pk\Big|_0^{2\pi}. \tag{13}$$

Lemma 3 *[8] If $u(k)$ is a bounded function on $[a,b]$ and $\ell = \frac{b-a}{M}$, then we have*

$$\Delta_\ell^{-1}u(k)\Big|_a^b = \sum_{r=1}^{M} u(b-r\ell) = \sum_{r=0}^{M-1} u(a+r\ell). \tag{14}$$

In general, when $k \in (\ell,\infty)$, we express

$$\Delta_\ell^{-1}u(k)\Big|_j^k = \sum_{r=1}^{[\frac{k}{\ell}]} u(k-r\ell) = \sum_{r=0}^{[\frac{k}{\ell}]-1} u(j+r\ell),\ j = k-[k/\ell]\ell. \tag{15}$$

3 Discrete Orthogonal System and Finite Fourier Series

Since a finite Fourier series is described by a family of discrete orthonormal functions, we introduce a orthogonal and further a orthonormal system of functions by defining the discrete inner product.

Definition 2 *Let $u(k)$ and $v(k)$ be bounded functions defined on $[a,b]$ and $\ell = \frac{b-a}{M}$. The discrete inner product of u and v with respect to ℓ is defined as*

$$(u,v)_\ell = \ell\Delta_\ell^{-1}u(k)v^*(k)\Big|_a^b = \ell\sum_{r=0}^{M-1} u(a+r\ell)v^*(a+r\ell) \tag{16}$$

and the discrete ℓ−norm of u, denoted by $\|u\|_{(\ell)}$ is defined as

$$\|u\|_{(\ell)} = (u,u)_\ell^{1/2} = \left\{\ell\Delta_\ell^{-1}|u(k)|^2\Big|_a^b\right\}^{1/2} = \left\{\ell\sum_{r=0}^{M-1}|u(a+r\ell)|^2\right\}^{1/2}. \tag{17}$$

Definition 3 *Let $I = [a,\ b]$, $\ell = \frac{b-a}{M}$ and $S_\ell = \{\phi_0,\phi_1,\phi_2,...\phi_M\}$ be a collection of bounded complex valued functions defined on I. If $(\phi_n,\phi_m)_\ell = 0$ whenever $m \neq n$, the collection S_ℓ is said to be a discrete orthogonal system, if in addition $\|\phi_n\|_\ell = 1$ for each n, then S_ℓ is said to be discrete orthonormal.*

Definition 4 *Let $S_\ell = \{\phi_0, \phi_1, \phi_2, ...\phi_M\}$ be an orthonormal on $I = [a,b]$, $\ell = \dfrac{b-a}{M}$, $u(k)$ is a bounded function on I and $c_n = (u, \phi_n)_\ell$, (say finite Fourier coefficients). Then the finite Fourier series of $u(k)$ related to S_ℓ is defined as*

$$u(k) = \sum_{n=0}^{M} c_n \phi_n(k), \;\; k \in \Big\{a + r\ell\Big\}_{r=0}^{M-1}. \tag{18}$$

Example 1 *Let $I = [a, a+2\pi]$, $\ell = \dfrac{\pi}{N}$ (here $M = 2N$) and*

$$\phi_0(k) = \frac{1}{\sqrt{2\pi}}, \; \phi_{2n-1}(k) = \frac{\sin nk}{\sqrt{\pi}}, \; \phi_{2n}(k) = \frac{\cos nk}{\sqrt{\pi}}, \; n = 1, 2, \cdots, N. \tag{19}$$

By (13) and $\Delta_\ell^{-1} k_\ell^{(0)}\Big|_a^{a+2\pi} = \dfrac{2\pi}{\ell}$, we find that $S_\ell = \{\phi_0, \phi_1, \phi_2, ...\phi_{2N}\}$ is a system of discrete orthonormal functions on I.
From (18), the finite Fourier series related to (19) is given by

$$u(k) = \frac{a_0}{2} + \sum_{n=1}^{N-1} (a_n \cos nk + b_n \sin nk) + \frac{a_N}{2} \cos Nk, \; k \in \Big\{a + r\ell\Big\}_{r=0}^{2N-1}, \tag{20}$$

where $a_n = \dfrac{\ell}{\pi} \Delta_\ell^{-1} (u(k) \cos nk)\Big|_a^{a+2\pi}$, $b_n = \dfrac{\ell}{\pi} \Delta_\ell^{-1} u(k) \sin nk\Big|_a^{a+2\pi}$ are got by (14).

Theorem 3.1 *Let $u(k) = \sum\limits_{n=0}^{M} c_n \phi_n(k)$, $k \in \Big\{a + r\ell\Big\}_{r=0}^{M-1}$ be the finite Fourier series of $u(k)$ relative to a discrete orthonormal set S_ℓ. Then we have*

$$\sum_{n=0}^{M} |c_n|^2 = \|u\|_{(\ell)}^2 \;\; (Discrete\ Parseval's\ Formula) \tag{21}$$

Proof Since S_ℓ is orthonormal and Δ_ℓ^{-1} is linear, (21) follows from (16), (17), (18) and the Definition 3.

□

Theorem 3.2 *Let $k \in (-\infty, \infty)$ and $\ell > 0$. If $n\ell \neq 2m\pi$, then we have*

$$\Delta_\ell^{-1} k_\ell^{(m)} \cos nk = \sum_{t=0}^{m} \sum_{r=0}^{t+1} \binom{t+1}{r} \frac{(m)_1^{(t)} \ell^t k_\ell^{(m-t)} \cos n(k - \ell + r\ell)}{(-1)^{r-1} \big(2(\cos n\ell - 1)\big)^{t+1}} \tag{22}$$

and

$$\Delta_\ell^{-1} k_\ell^{(m)} \sin nk = \sum_{t=0}^{m} \sum_{r=0}^{t+1} \binom{t+1}{r} \frac{(m)_1^{(t)} \ell^t k_\ell^{(m-t)} \sin n(k - \ell + r\ell)}{(-1)^{r-1} \big(2(\cos n\ell - 1)\big)^{t+1}}. \tag{23}$$

Proof Taking $u(k)=k_\ell^{(1)}$, $w(k)=\cos nk$ in (12) and using (9) and (11), we get
$\Delta_\ell^{-1}\Big(k_\ell^{(1)}\cos nk\Big)=k_\ell^{(1)}\Big(\frac{\cos n(k-\ell)-\cos nk}{2(1-\cos n\ell)}\Big)-\Delta_\ell^{-1}\Big(\frac{\cos nk-\cos n(k+\ell)}{2(1-\cos n\ell)}\ell\Big).$
Since Δ_ℓ^{-1} is linear, applying (11) for both $\cos nk$ and $\cos n(k+\ell)$, we get

$$\Delta_\ell^{-1}\Big(k_\ell^{(1)}\cos nk\Big)=k_\ell^{(1)}\Big(\frac{\cos n(k-\ell)-\cos nk}{2(1-\cos n\ell)}\Big)-\frac{\ell\Big(\cos n(k-\ell)-2\cos nk+\cos n(k+\ell)\Big)}{(2(1-\cos n\ell))^2}. \tag{24}$$

Taking $u(k)=k_\ell^{(2)}$, $w(k)=\cos nk$ in (12), and using (9), (11) and (24), we get

$$\Delta_\ell^{-1}\Big(k_\ell^{(2)}\cos nk\Big)=\frac{k_\ell^{(2)}(\cos n(k-\ell)-\cos nk)}{2(1-\cos n\ell)}-\frac{2\ell k_\ell^{(1)}\Big(\cos n(k-\ell)-2\cos nk+\cos n(k+\ell)\Big)}{(2(1-\cos n\ell))^2}+\frac{(1\ell)(2\ell)\Big(\cos n(k-\ell)-3\cos nk+3\cos n(k+\ell)-\cos n(k+2\ell)\Big)}{(2(1-\cos n\ell))^3} \tag{25}$$

which can be expressed as
$\Delta_\ell^{-1}k_\ell^{(2)}\cos nk=\sum\limits_{t=0}^{2}\sum\limits_{r=0}^{t+1}\binom{t+1}{r}\frac{(2)_1^{(t)}\ell^t k_\ell^{(2-t)}\cos n(k-\ell+r\ell)}{(-1)^{r-1}\big(2(\cos n\ell-1)\big)^{t+1}}.$
Continuing the above process, we get the relation (22).
Now, (23) follows by replacing $\cos nk$ by $\sin nk$ in (22).

□

Corollary 1 *When $I=[0,\ 2\pi]$, $\ell=\dfrac{\pi}{N}$, $k\in\Big\{r\ell\Big\}_0^{2N-1}$, the finite Fourier coefficients a_n and b_n for the polynomial factorial $k_\ell^{(m)}$ are given by*

$$a_0=\frac{\ell}{\pi}\Delta_\ell^{-1}k_\ell^{(m)}\Big|_0^{2\pi}=\frac{(2\pi)_\ell^{(m+1)}}{\pi(m+1)}, \tag{26}$$

$$a_n=\frac{\ell}{\pi}\Delta_\ell^{-1}k_\ell^{(m)}\cos nk\Big|_0^{2\pi}=\sum_{t=0}^{m-1}\sum_{r=0}^{t+1}\binom{t+1}{r}\frac{(m)_1^{(t)}\ell^t(2\pi)_\ell^{(m-t)}\cos n(r-1)\ell}{N(-1)^{r-1}\big(2(\cos n\ell-1)\big)^{t+1}} \tag{27}$$

and

$$b_n=\frac{\ell}{\pi}\Delta_\ell^{-1}k_\ell^{(m)}\sin nk\Big|_0^{2\pi}=\sum_{t=0}^{m-1}\sum_{r=0}^{t+1}\binom{t+1}{r}\frac{(m)_1^{(t)}\ell^t(2\pi)_\ell^{(m-t)}\sin n(r-1)\ell}{N(-1)^{r-1}\big(2(\cos n\ell-1)\big)^{t+1}}. \tag{28}$$

Proof The proof follows by applying the limit 0 to 2π in (22) and (23), and then multiplying by ℓ/π.

□

Example 2 *From (20), and using (26), (27) and (28) for a_0, a_n and b_n respectively, we get the finite Fourier series for the polynomial factorial $k_\ell^{(m)}$ as*

$$k_\ell^{(m)} = \frac{a_0}{2} + \sum_{n=1}^{N-1}(a_n\cos nk + b_n\sin nk) + \frac{a_N}{2}\cos Nk,\ k \in \left\{r\ell\right\}_{r=0}^{2N-1}. \quad (29)$$

In particular, when $N = 20$, $\ell = \dfrac{\pi}{20}$ and $m = 15$, (29) becomes,

$$(k)_{\pi/20}^{(15)} = \frac{(2\pi)_{\pi/20}^{16}}{32\pi} + \sum_{n=1}^{19}(a_n\cos nk + b_n\sin nk) + \frac{a_{20}}{2}\cos 20k,\ k \in \left\{\frac{r\pi}{20}\right\}_{r=0}^{39}.$$

Theorem 3.3 *Let $k \in (-\infty,\infty)$, $\ell > 0$ and $n\ell \neq m2\pi$, then we have*

$$\Delta_\ell^{-1}k^p\cos nk = \sum_{m=1}^{p}\sum_{t=0}^{m}\sum_{r=0}^{t+1}\binom{t+1}{r}\frac{S_m^p(m)_1^{(t)}k_\ell^{(m-t)}\cos n(k-\ell+r\ell)}{(-1)^{r-1}\ell^{m-t-p}\big(2(\cos n\ell-1)\big)^{t+1}} \quad (30)$$

and

$$\Delta_\ell^{-1}k^p\sin nk = \sum_{m=1}^{p}\sum_{t=0}^{m}\sum_{r=0}^{t+1}\binom{t+1}{r}\frac{S_m^p(m)_1^{(t)}k_\ell^{(m-t)}\sin n(k-\ell+r\ell)}{(-1)^{r-1}\ell^{m-t-p}\big(2(\cos n\ell-1)\big)^{t+1}}. \quad (31)$$

Proof The proof follows by second term of (8) and applying (22).

□

Corollary 2 *When $I = [0,\ 2\pi]$, $\ell = \dfrac{\pi}{N}$, the finite Fourier coefficients a_n and b_n for $n = 0,1,2,\cdots,N$ for polynomial k^p are given by*

$$a_n = \frac{\ell}{\pi}\Delta_\ell^{-1}k^p\cos nk\Big|_0^{2\pi} = \sum_{m=1}^{p-1}\sum_{t=0}^{m}\sum_{r=0}^{t+1}\binom{t+1}{r}\frac{S_m^p(m)_1^{(t)}(2\pi)_\ell^{(m-t)}\cos n(r-1)\ell}{(-1)^{r-1}N\ell^{m-t-p}\big(2(\cos n\ell-1)\big)^{t+1}} \quad (32)$$

and

$$b_n = \frac{\ell}{\pi}\Delta_\ell^{-1}k^p\sin nk\Big|_0^{2\pi} = \sum_{m=1}^{p-1}\sum_{t=0}^{m}\sum_{r=0}^{t+1}\binom{t+1}{r}\frac{S_m^p(m)_1^{(t)}(2\pi)_\ell^{(m-t)}\sin n(r-1)\ell}{(-1)^{r-1}N\ell^{m-t-p}\big(2(\cos n\ell-1)\big)^{t+1}}. \quad (33)$$

Proof The proof follows by applying the limits 0 to 2π in (30) and (31), and then multiplying by ℓ/π.

□

Corollary 3 *From (20), and using (32) and (33) for a_n and b_n respectively, we get the finite Fourier series for the polynomial k^p as*

$$k^p = \frac{a_0}{2} + \sum_{n=1}^{N-1}(a_n \cos nk + b_n \sin nk) + \frac{a_N}{2}\cos Nk,\ k \in \Big\{r\ell\Big\}_{r=0}^{2N-1}. \tag{34}$$

Corollary 4 *Let $I=[0,2\pi]$, $\ell = \frac{\pi}{N}$, and $c>0$ be a constant. Then, for $k \in \Big\{r\ell\Big\}_{r=0}^{2N-1}$, the finite Fourier series of the geometric function c^k is given by*

$$c^k = \frac{\ell}{2\pi}\Big(\frac{c^{2\pi}-1}{c^{\ell}-1}\Big) + \frac{\ell}{\pi}\sum_{n=1}^{N-1}\Big(\cos nk \sum_{r=1}^{[2\pi/\ell]} c^{(2\pi-r\ell)}\cos nr\ell - \sin nk \sum_{r=1}^{[2\pi/\ell]} c^{(2\pi-r\ell)}\sin nr\ell\Big) + \frac{\ell}{2\pi}\cos Nk \sum_{r=1}^{[2\pi/\ell]} c^{(2\pi-r\ell)}\cos Nr\ell. \tag{35}$$

Proof The proof follows by taking $u(k)=c^k$ in (20), and then applying (15).

□

The following example is a verification of Corollary 4

Example 3 *Taking $c=8$, $N=100$ and $\ell=\frac{\pi}{100}$ in (35), we have*

$$8^k = \frac{8^{2\pi}-1}{200(8^{\pi/100}-1)} + \frac{1}{100}\sum_{n=1}^{99}\Big(\sum_{r=1}^{200} 8^{(2\pi-r(\pi/100))}\cos\frac{nr\pi}{100}\cos nk - \sum_{r=1}^{200} 8^{(2\pi-r(\pi/100))}\sin\frac{nr\pi}{100}\sin nk\Big) + \frac{1}{200}\sum_{r=1}^{200}\cos r\pi\cos 100k,\ k \in \Big\{\frac{r\pi}{100}\Big\}_{r=0}^{199}.$$

Theorem 3.4 *Let $I=[2\pi,4\pi]$, $m \in (-\infty,\infty)$, $\ell=\frac{\pi}{N}$. Then the finite Fourier series of rational function $\frac{1}{k^m}$ is given by*

$$\frac{1}{k^m} = \frac{\ell}{2\pi}\sum_{r=1}^{[2\pi/\ell]}\frac{1}{(4\pi-r\ell)^m} + \frac{\ell}{\pi}\sum_{n=1}^{N-1}\Big(\sum_{r=1}^{[2\pi/\ell]}\frac{\cos nr\ell}{(4\pi-r\ell)^m}\cos nk - \sum_{r=1}^{[2\pi/\ell]}\frac{\sin nr\ell}{(4\pi-r\ell)^m}\sin nk\Big) + \frac{\ell}{2\pi}\sum_{r=1}^{[2\pi/\ell]}\frac{\cos Nr\ell}{(4\pi-r\ell)^m}\cos Nk,\ k \in \Big\{2\pi+r\ell\Big\}_{r=0}^{2N-1}. \tag{36}$$

Proof Taking $a=2\pi$, $u(k)=\frac{1}{k^m}$ in (20) and applying (15), we get (36).

□

The following example is a verification of the Corollary 3.4.

Example 4 *Taking* $m = 5$, $N = 24$, $\ell = \dfrac{\pi}{24}$ *in (36), then we have*

$$\frac{1}{k^5} = \frac{1}{48}\sum_{r=1}^{48}\frac{1}{(4\pi - r(\pi/24))^5} + \frac{1}{24}\sum_{n=1}^{23}\Big(\sum_{r=1}^{48}\frac{\cos nr(\pi/24)}{(4\pi - r(\pi/24))^5}\cos nk - \sum_{r=1}^{48}\frac{\sin nr(\pi/24)}{(4\pi - r(\pi/24))^5}\sin nk\Big) + \frac{1}{48}\sum_{r=1}^{48}\frac{\cos r\pi}{(4\pi - r(\pi/24))^5}\cos 24k,\ k \in \Big\{2\pi + \frac{r\pi}{24}\Big\}_{r=0}^{47}$$

Theorem 3.5 *Let* $I = [\pi/8, 17\pi/8]$, $\ell = \dfrac{\pi}{N}$. *Then, the finite Fourier series of the logarithmic function* $\log k$ *is given by*

$$\log k = \frac{\ell}{2\pi}\sum_{r=1}^{[2\pi/\ell]}\log(\tfrac{17\pi}{8} - r\ell) + \frac{\ell}{\pi}\sum_{n=1}^{N-1}\Big(\sum_{r=1}^{[2\pi/\ell]}\log(\tfrac{17\pi}{8} - r\ell)\cos n(\tfrac{17\pi}{8} - r\ell)\cos nk + \sum_{r=1}^{[2\pi/\ell]}\log(\tfrac{17\pi}{8} - r\ell)\sin n(\tfrac{17\pi}{8} - r\ell)\sin nk\Big)$$
$$+\frac{\ell}{2\pi}\sum_{r=1}^{[2\pi/\ell]}\log\big(\frac{17\pi}{8} - r\ell\big)\cos N\big(\frac{17\pi}{8} - r\ell\big)\cos Nk,\ k \in \Big\{\frac{\pi}{8} + r\ell\Big\}_{r=0}^{2N-1}. \tag{37}$$

Proof The proof follows by taking $a = \dfrac{\pi}{8}$, $u(k) = \log k$ in (20) and applying (15).

□

The following example is a verification of the Corollary 3.5

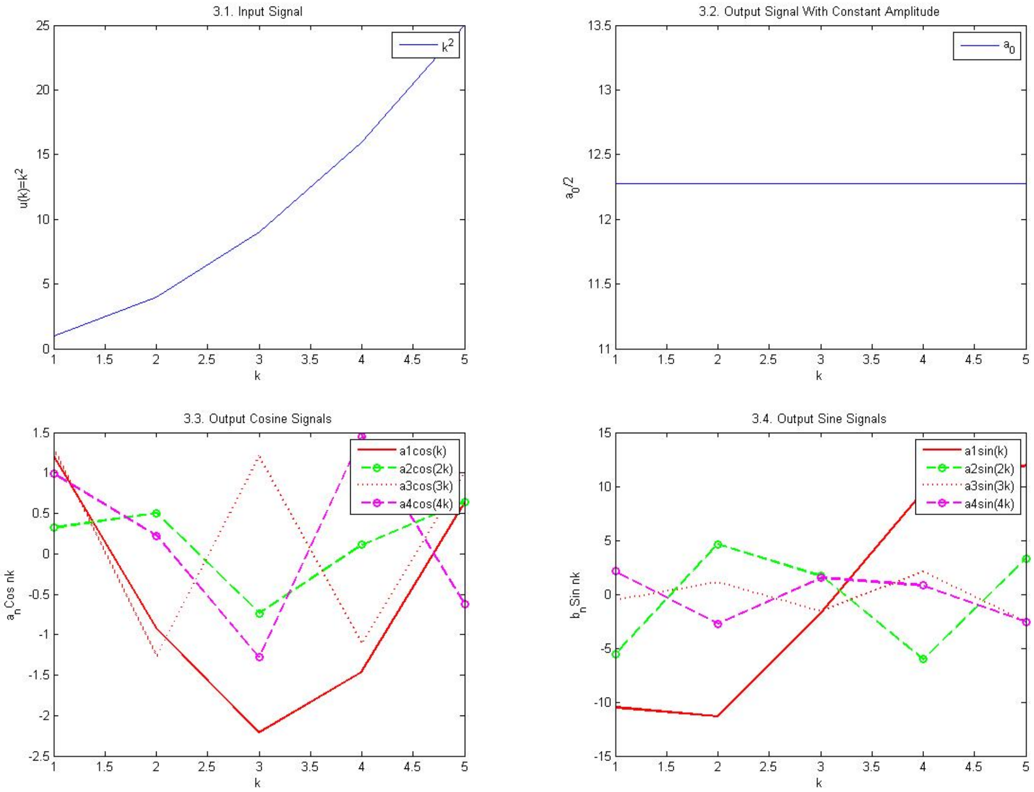
Example 5 *Taking* $N = 8$, $\ell = \dfrac{\pi}{8}$ *in (37), for* $k \in \Big\{\frac{\pi + r\pi}{8}\Big\}_{r=0}^{15}$, *we have*

$$\log k = \frac{1}{16}\sum_{r=1}^{16}\log(\tfrac{17\pi - r\pi}{8}) + \frac{1}{8}\sum_{n=1}^{7}\Big(\sum_{r=1}^{16}\log(\tfrac{17\pi - r\pi}{8})\cos n(\tfrac{17\pi - r\pi}{8})\cos nk + \sum_{r=1}^{16}\log(\tfrac{17\pi - r\pi}{8})\sin n(\tfrac{17\pi - r\pi}{8})\sin nk\Big) + \frac{1}{16}\sum_{r=1}^{16}\log(\tfrac{17\pi - r\pi}{8})\cos 8(\tfrac{17\pi - r\pi}{8})\cos 8k.$$

Taking $p = 2$, $N = 11$ *and* $\ell = \dfrac{\pi}{11}$ *in (34), we have*

$$k^2 = \frac{301\pi^2}{242} + \sum_{n=1}^{10}\Big(\frac{-2\pi^2}{121}\Big(11 - \frac{1}{1 - \cos n(\pi/11)}\Big)\cos nk + \frac{22\pi^2 \sin n(21\pi/11)}{121(1 - \cos n(\pi/11))}\sin nk\Big) - \frac{21\pi^2}{242}\cos 11k,\ k \in \Big\{\frac{r\pi}{11}\Big\}_{r=0}^{21}.$$

Here we provide MATLAB coding for verification FS: *syms n*
pi.^2 = ((301 ∗ pi.^2)./242) + symsum(((−2 ∗ pi.^2./121) ∗ (11 − 1./(1 − cos(n ∗ pi./11)))) ∗ cos(n ∗ pi) + ((22 ∗ pi.^2./121) ∗ (sin(n ∗ 21 ∗ pi./11)./(1 − cos(n ∗ pi./11)))) ∗ sin(n ∗ pi), n, 1, 10) − (2 ∗ pi.^2./242) ∗ (11 − 1./(1 − cos(pi))) ∗ cos(11 ∗ pi)



Discussion: The diagrams 3.2 to 3.4 give 9 components of the decomposition of the function $u(k) = k^2$ (input signal). One can get the remaining 13 components easily.

4 Conclusion:

The Fourier series and its transforms have wide range of applications specially in the field of digital signal process. For functions which have no usual Fourier series expression, we are able to find finite Fourier series expression (decomposition) using summation solution of generalized difference equation. The method discussed in this paper leads to several applications in signal process.

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