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Coupled Fixed Point Theorems in C*-Algebra-Valued b-Metric Spaces

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Abstract: In this paper, we give some coupled fixed point results in the framework of C^*-algebra-valued b-metric spaces and in particular in the setting of C^* algebra-valued metric spaces. These results, with shorter proofs, generalize and improve other theorems recently introduced. We have used a method of reducing coupled fixed point results to the respective ones for mappings with one variable in the framework of b-metric spaces. Finally, two examples are given to support our theoretical work.

Keywords: metric space, coupled fixed point.

1 Introduction and Preliminaries

The Banach contraction mapping principle is one of the most powerful and useful tools in modern mathematics and has many generalizations. Currently, extending Banach contraction mapping principle by replacing the metric space with a generalized one and extend theorem in this generalized structure are of great interest in mathematics. Many generalized metric spaces that have been introduced, have attracted many researching mathematics, for example we point out: b-metric space, cone metric space, cone valued metric space or tvs-cone metric space, cone b-metric space or cone metric type space, cone metric space over Banach algebra, cone b-metric space over Banach algebra, C^*-algebra-valued metric space, C^*-algebra-valued b-metric space, etc. Recently, using the notion and properties of C^*-algebra (see [15]), several authors ([1], [2], [6], [10], [11], [12], [13], [14], [16], [18], [19]) have introduced and considered the concepts of C^*-algebra-valued metric spaces as well as C^*-algebra-valued b-metric spaces and have given some fixed point theorems for self-mappings satisfying contractive conditions on such spaces. It is possible to find some results concerning b-metric spaces in [3, 5].

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In this paper, some recent coupled fixed point results established for a complete C^*-algebra valued b-metric space are generalized and improved, with much shorter proofs. We have used a method to reducing coupled fixed point results to the respective ones for mappings with one variable in the framework of b-metric spaces. Also, we have used the fact that each C^*-algebra-valued b-metric space is a cone b-metric space over normal cone with normal constant equal to 1.

Firstly, we begin with the basic concept of C^*-algebras. A real or a complex linear space $\mathscr{A}$ is an algebra if vector multiplication is defined for every pair of elements of $\mathscr{A}$ such that $\mathscr{A}$ is a ring with respect to both vector addition and vector multiplication and for every scalar β and every pair of elements $u,v \in \mathscr{A}$, we have $\beta(uv) = (\beta u)v = u(\beta v)$. If $\mathscr{A}$ is endowed with a submultiplicative norm $\|\cdot\|$, that is, $\|uv\| \leq \|u\|\|v\|$ for all $u,v \in \mathscr{A}$, then $(\mathscr{A}, \|\cdot\|)$ is a normed algebra. A complete normed algebra is called Banach algebra. An involution on the algebra $\mathscr{A}$ is a conjugate linear mapping $* : \mathscr{A} \to \mathscr{A}$ such that

(1) $u^{**} = u$;

(2) $(uv)^* = v^*u^*$

for all $u,v \in \mathscr{A}$. The pair $(\mathscr{A}, *)$ is called $*$-algebra. A Banach $*$-algebra $\mathscr{A}$ is a $*$-algebra with a complete submultiplicative norm such that $\|u^*\| = \|u\|$ for all $u \in \mathscr{A}$. Then, a C^*-algebra is a Banach $*$-algebra such that $\|u^*u\| = \|u\|^2$. The set $\mathbb{C}$ of complex numbers, the set $L(H)$ of all bounded linear operators on a Hilbert space H, and the set $M_n(\mathbb{C})$ of $n \times n$-matrices are examples of C^*-algebras. If a normed algebra $\mathscr{A}$ admits a unit I, that is, there exists an element $I \in \mathscr{A}$ such that $Iu = uI = u$ for all $u \in \mathscr{A}$, and $\|I\| = 1$, we say that $\mathscr{A}$ is an unital normed algebra. A complete unital normed algebra $\mathscr{A}$ is called unital Banach algebra. In this paper, we will assume that $\mathscr{A}$ is an unital C^*-algebra with a unit I. For the basic properties and results in the setting of C^*-algebras, the interested reader is referred to [15] and the references therein.

Let $\mathscr{A}$ be a C^*-algebra. An element $a \in \mathscr{A}$ is called positive if $a = a^*$ and the spectrum $\sigma(a)$ of a is a subset of nonnegative real numbers. The set of positive elements in $\mathscr{A}$ is denoted by $\mathscr{A}_+$. We define an order relation $\preceq$ by using $\mathscr{A}_+$, where $a \preceq b$ if $a = b$ or $b - a$ is a positive element. We use the notation $\theta \preceq a$ to denote that a is a positive element, where θ is the zero element in $\mathscr{A}$. Now, we recall some properties of the elements of $\mathscr{A}_+$.

(j) The set $\mathscr{A}_+ = \{a^*a : a \in \mathscr{A}\}$ is a closed cone in $\mathscr{A}$;

(jj) if $\theta \preceq a \preceq b$, then $\|a\| \leq \|b\|$;

(jjj) if $\theta \preceq a \preceq b$, then $\theta \preceq \lambda^* a \lambda \preceq \lambda^* b \lambda$ for all $\lambda \in \mathscr{A}$;

(jv) if $a,b \in \mathscr{A}_+$ and $ab = ba$; then $\theta \preceq ab$;

The concept of C^*-algebra-valued b-metric space was introduced by Ma and Jiang [14] as follows.

Definition 1. *Let X be a nonempty set. A mapping $d_b : X \times X \to \mathscr{A}_+$ is called C^*-algebra-valued b-metric on X if there exists $b \in \mathscr{A}$, with $I \preceq b$ and $ab = ba$ for all $a \in \mathscr{A}$, such that the following conditions hold:*

(1) $\theta = d_b(u,v)$ *if and only if* $u = v$*;*

(2) $d_b(u,v) = d_b(v,u)$ *for all* $u,v \in X$*;*

(3) $d_b(u,v) \preceq b(d_b(u,z) + d_b(z,v))$ *for all* $u,v,z \in X$.

Then $(X, \mathscr{A}, d_b)$ is called a C^*-algebra-valued b-metric space.

If $b = I$, from Definition 1 we obtain the concept of C^*-algebra-valued metric introduced by Ma et al. [13]. In this case use d to denote the C^*-algebra-valued metric and $(X, \mathscr{A}, d)$ is called a C^*-algebra-valued metric space.

Let $(X, \mathscr{A}, d_b)$ be a C^*-algebra-valued b-metric space and $\{p_n\} \subset X$ be a sequence.

(i) $\{p_n\}$ is called *convergent* to $p \in X$, written as $\lim_{n\to\infty} p_n = p$, if $\lim_{n\to\infty} \|d_b(p_n, p)\| = 0$.

(ii) $\{p_n\}$ is called Cauchy if $\lim_{n,m\to\infty} \|d_b(p_n, p_m)\| = 0$.

(iii) $(X, \mathscr{A}, d_b)$ is called complete if each Cauchy sequence is a convergent sequence.

The following remark is used to obtain fixed point results in C^*-algebra-valued b-metric spaces.

Remark 2. *Every C^*-algebra-valued b-metric on a set X induces on X a b-metric D_b with constant $\|b\|$, where $D_b : X \times X \to [0,\infty)$ is defined by $D_b(u,v) = \|d(u,v)\|$ for all $u,v \in X$. To verify that D_b is a b-metric is sufficient to show that the triangular inequality holds. By using (jj), we get*

$$\begin{aligned} D_b(u,v) = \|d(u,v)\| &\le \|b(d(u,z) + d(z,v))\| \\ &\le \|b\|(\|d(u,z)\| + \|d(z,v)\|) \\ &= \|b\|(D_b(u,z) + D_b(z,v)). \end{aligned}$$

If $b = I$, then we use the notation D instead of D_b. In this case D is a metric on X. Also, note that if $(X, \mathscr{A}, d_b)$ is a complete C^*-algebra-valued b-metric space, then $(X, D_b, \|b\|)$, is a complete b-metric space.

Ma et al. proved this main result.

Theorem 3 ([13], Theorem 2.1)**.** *Let $(X,\mathscr{A},d)$ be a complete C^*-algebra-valued metric space, and let $T:X\to X$ be a mapping. Assume that there exists $\alpha\in\mathscr{A}$, with $\|\alpha\|<1$, such that*

$$d(Tu,Tv)\preceq\alpha^*d(u,v)\alpha,\quad \text{for all } u,v\in X. \tag{1}$$

Then T has a unique fixed point in X.

Now, using Remark 2 with $b=I$, we deduce that Theorem 2.1 of [13] is a consequence of Banach contraction principle (see [10], Theorem 2).

2 Main results

In the following part of our paper we consider, extend, generalize, unify, improve and enrich some results of coupled fixed point in the framework of C^*-algebra-valued b-metric space. Firstly, we note that if $(X,\mathscr{A},d_b)$ is a complete C^*-algebra-valued b-metric space, then $(X\times X,\mathscr{A},d_{b+})$ is also a complete C^*-algebra-valued b-metric space, where $d_{b+}(Y,V)=d_{b^+}((x,y),(u,v))=d_b(x,u)+d_b(y,v)$, for all $Y=(x,y),V=(u,v)\in X\times X$. Also, to each mapping $T:X\times X\to X$, we associate the mapping $G_T:X\times X\to X\times X$ defined by $G_T(Y)=G_T(x,y)=(T(x,y),T(y,x))$ for all $Y(x,y)\in X\times X$. A point $(x,y)\in X\times X$ is a fixed point of T, if $T(x,y)=x$ and $T(y,x)=y$. Note that T has a unique coupled fixed point if and only if G_T has a unique fixed point.

We recall some results in the setting of b-metric space.

Lemma 4 ([9], Corollary 3.9)**.** *Let (X,D,b) be a complete b-metric space, and let $T:X\to X$ satisfy*

$$D(Tu,Tv)\le a_1D(u,v)+a_2D(u,Tu)+a_3D(v,Tv)+a_4D(u,Tv)+a_5D(v,Tu) \tag{2}$$

for all $u,v\in X$ where a_i, $(i=1,...,5)$, are nonnegative constant such that

$$2ba_1+(b+1)(a_2+a_3)+(b^2+b)(a_4+a_5)<2.$$

Then T has a unique fixed point.

Lemma 5 ([5], Theorem 2.1)**.** *Let (X,D,b) be a complete b-metric space, and let $T:X\to X$ be a mapping such that for some $\lambda\in[0,1)$*

$$D(Tu,Tv)\le\lambda D(u,v) \tag{3}$$

holds for all $u,v\in X$. Then T has a unique fixed point z, and for every $u_0\in X$, the sequence $\{T^nu_0\}$ converges to z.

Proof. Firstly, if $\lambda \in [0, \frac{1}{b})$ the proof follows from Lemma 4 with $a_1 = \lambda$ and $a_2 = a_3 = a_4 = a_5 = 0$. Therefore, let $\lambda \in [\frac{1}{b}, 1)$. It is clear that (3) implies

$$D(T^n u, T^n v) \leq \lambda^n D(u, v), \tag{4}$$

for all $n \in \mathbb{N}$. Since $\lambda^n \to 0$ as $n \to \infty$ we get that there exists $k \in \mathbb{N}$ such that $\lambda^k < \frac{1}{b}$. Now, again according to Lemma 4 we obtain that T^k, has a unique fixed point (say z). Consequently, z is a unique fixed point of T. The proof of Lemma 5 is complete. □

The following theorem is a Banach type result of coupled fixed point that generalizes Theorem 2.1 of [2].

Theorem 6. *Let $(X, \mathscr{A}, d_b)$ be a complete C^*-algebra-valued b-metric space. Suppose that the mapping $T : X \times X \to X$ satisfies the following condition:*

$$d_b(T(z,w), T(u,v)) + d_b(T(w,z), T(v,u)) \preceq 2[\alpha^* d_b(z,u)\alpha + \alpha^* d_b(w,v)\alpha], \tag{5}$$

for any $z, w, u, v \in X$, where $\alpha \in \mathscr{A}$ with $2\|\alpha\|^2 < 1$. Then T has a unique coupled fixed point.

Proof. It is clear that the mapping F has a unique coupled fixed point if and only if the mapping $G_T : X \times X \to X \times X$, where $G_T(Y) = G_T(x,y) = (T(x,y), T(y,x))$, has a unique fixed point. Now, the condition (5) implies that the relation

$$d_{b^+}(G_T(Y), G_T(V)) \preceq \left(\sqrt{2}a^*\right) d_{b^+}(Y, V) \left(\sqrt{2}a\right), \tag{6}$$

holds for all $Y = (z,w), V = (u,v) \in X \times X$, where $\alpha \in \mathscr{A}$ with $2\|\alpha\|^2 < 1$. Using the condition (jj), from (6) we get

$$D_b(G_T Y, G_F V) \leq 2\|\alpha\|^2 D_b(Y, V),$$

for all $Y, V \in X \times X$. Since $(X \times X, D_b, \|b\|)$ is a complete b-metric space the result follows by Lemma 5 with $\lambda = 2\|\alpha\|^2$. □

Remark 7. *If in the previous Theorem 6 we suppose that $b = I$, then we obtain a result of existence of a unique fixed point in the setting of C^*-algebra-valued metric space.*

From Theorem 6 and Remark 7, we deduce the following result.

Corollary 8 ([19], Theorem 2.1)**.** *Let $(X, \mathscr{A}, d)$ be a complete C^*-algebra-valued metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following condition:*

$$d(F(z,w), F(u,v)) \preceq \alpha^* d(z,u)\alpha + \alpha^* d(w,v)\alpha, \tag{7}$$

for any $z, w, u, v \in X$, where $\alpha \in \mathscr{A}$ with $2\|\alpha\|^2 < 1$. Then F has a unique coupled fixed point.

Proof. For all $z,w,u,v\in X$, from (7), we have

$$d\left(F\left(z,w\right),F\left(u,v\right)\right)\preceq\alpha^{*}d\left(z,w\right)\alpha+\alpha^{*}d\left(u,v\right)\alpha$$

and

$$d\left(F\left(w,z\right),F\left(v,u\right)\right)\preceq\alpha^{*}d\left(w,z\right)\alpha+\alpha^{*}d\left(v,u\right)\alpha.$$

It implies that

$$d\left(F\left(z,w\right),F\left(u,v\right)\right)+d\left(F\left(w,z\right),F\left(v,u\right)\right)\preceq\left(\sqrt{2}\alpha^{*}\right)\left(d\left(z,w\right)+d\left(u,v\right)\right)\left(\sqrt{2}\alpha\right).$$

Therefore, (7) implies (5) and we have that our Theorem 6 generalizes Theorem 2.1 of [2] and if $b=I$ Theorem 2.1 of [19]. □

The following example shows that Theorem 6 (with $b=I$) is a genuine generalization of Theorem 2.1 of [19].

Example 9. *Let $\mathscr{A}=M_{2\times2}(\mathbb{R})$ endowed with the norm $\|A\|=\max_{i,j}\left|a_{ij}\right|$, where a_{ij} are the entries of the matrix $A\in M_{2\times2}(\mathbb{R})$, and the involution given by $A^{*}=\left(\overline{A}\right)^{T}=A^{T}$. Clearly, each matrix of type $A=\begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix}\in\mathscr{A}_{+}$ if $\alpha,\beta\geq0$. This implies that $\begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix}\preceq\begin{bmatrix}\delta & 0\\ 0 & \gamma\end{bmatrix}$ if and only if $\alpha\leq\delta$ and $\beta\leq\gamma$.*

Let $X=\mathbb{R}$, and

$$d\left(u,v\right)=\begin{bmatrix}|u-v| & 0\\ 0 & |u-v|\end{bmatrix}=|u-v|\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=|u-v|I. \qquad (8)$$

Clearly, $(X,\mathscr{A},d)$ is a complete C^{*}-algebra-valued metric space. We define a mapping $F:X\times X\to X$ by $F(z,w)=\frac{z-2w}{7}$. We say that (7) implies (5). We claim that F satisfies condition (5) with respect to $\alpha=\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix}\in\mathscr{A}$, but does not satisfy (7). Indeed, let us assume that F satisfies (7), that is,

$$\begin{aligned}|F(z,w)-F(u,v)|I &\preceq \alpha^{*}|z-u|I\alpha+\alpha^{*}|w-v|I\alpha\\ &\preceq \alpha^{*}\left(|z-u|+|w-v|\right)I\alpha.\end{aligned}$$

By using (jj), from the previous inequality, we get

$$\left|\frac{z-u-2\left(w-v\right)}{7}\right|\leq\frac{1}{4}\left(|z-u|+|w-v|\right),$$

for all $z,w,u,v \in X$. *Putting* $z = u, w \neq v$ *we get* $\frac{2}{7} \leq \frac{1}{4}$. *A contradiction.*

Now, we prove that (5) *holds, that is,*

$$|F(z,w) - F(u,v)|I + |F(w,z) - F(v,u)|I \preceq \left(\sqrt{2}\alpha^*\right)(|z-u| + |w-v|)I\left(\sqrt{2}\alpha\right),$$

for all $z,w,u,v \in X$ *or equivalently,*

$$\left|\frac{z-u-2(w-v)}{7}\right| + \left|\frac{w-v-2(z-u)}{7}\right| \leq \frac{1}{2}(|z-u| + |w-v|).$$

However,

$$\left|\frac{z-u-2(w-v)}{7}\right| + \left|\frac{w-v-2(z-u)}{7}\right| \leq \frac{3}{7}(|z-u| + |w-v|),$$

for all $z,w,u,v \in X$, *so the result follows from the fact that* $\frac{3}{7} < \frac{1}{2}$.

Hence, all the conditions of Theorem 6 with $b = I$ *are satisfied. This means that* F *has a unique coupled fixed point* $(0,0)$, *but Theorem 2.1 of [19] cannot be applied to* F *in this example.*

Example 10. *Let* $\mathscr{A}$ *be the* C^**- algebra considered in Example 9,* $X = \mathbb{R}$, $b = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ *and*

$$d(z,w) = \begin{bmatrix} |z-w|^2 & 0 \\ 0 & |z-w|^2 \end{bmatrix} = |z-w|^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |z-w|^2 I,$$

Then $(X,\mathscr{A},d)$ *is a* C^**-algebra-valued b-metric space. Let us define a mapping* $F : X \times X \to X$ *by* $F(z,w) = \frac{z}{\sqrt{2}}$ *for all* $z,w \in X$ *and take* $\alpha = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ *where* $0 < k < \frac{1}{\sqrt{2}}$. *Now,* F *does not satisfy* (7). *Indeed, let us suppose that* (7) *holds, that is*

$$\left|\frac{z-u}{\sqrt{2}}\right|^2 \leq k^2\left(|z-u|^2 + |w-v|^2\right),$$

for all $z,w,u,v \in X$. *Putting* $z \neq u, w = v$ *we get* $\frac{1}{\sqrt{2}} \leq k$, *which is contradiction.*

Now, we shall prove that F *satisfies condition* (5), *that is., for all* $z,w,u,v \in X$ *holds:*

$$\left|\frac{z-u}{\sqrt{2}}\right|^2 + \left|\frac{w-v}{\sqrt{2}}\right|^2 \leq 2k^2\left(|z-u|^2 + |w-v|^2\right).$$

Hence, the last inequality holds for all $z,w,u,v \in X$ *if and only if* $\frac{1}{2} \leq k$. *Since* $\frac{1}{2} < \frac{1}{\sqrt{2}}$ *there exists* $k \in [\frac{1}{2}, \frac{1}{\sqrt{2}})$ *such that the condition* (5) *holds while* (7) *does not satisfy. Hence, all the conditions of Theorem 6 are satisfied. This means that* F *has a unique coupled fixed point, but Theorem 2.1 of [2] cannot be applied to* F *in this example.*

The following is a Kannan type result of coupled fixed point.

Theorem 11. *Let $(X, \mathscr{A}, d_b)$ be a complete C^*-algebra-valued b-metric space. Suppose that the mapping $T : X \times X \to X$ satisfies the following condition*

$$d_b\left(T(z,w), T(u,v)\right) + d_b\left(T(w,z), T(v,u)\right)$$

$$\preceq \alpha\left(d_b\left(T(z,w), z\right) + d_b\left(T(w,z), w\right)\right) + \beta\left(d_b\left(T(u,v), u\right) + d_b\left(T(v,u), v\right)\right), \tag{9}$$

for any $z,w,u,v \in X$, where $\alpha, \beta \in \mathscr{A}$ with $2\|b\|\,\|\alpha\| + (\|b\|+1)\|\beta\| < 2$. Then T has a unique coupled fixed point.

Proof. We say that the mapping F has a unique coupled fixed point if and only if the mapping $G_T : X \times X \to X \times X$, where $G_T(Y) = G_T(x,y) = (T(x,y), T(y,x))$, has a unique fixed point. Now, the condition (9) implies that the relation

$$d_{b^+}\left(G_T(Y), G_T(V)\right) \preceq \alpha d_{b+}\left(G_T(Y), Y\right) + \beta d_{b^+}\left(G_T(V), V\right), \tag{10}$$

holds for all $Y, V \in X \times X$. This means that G_T is a Kannan type contractive mapping in the framework of C^*-algebra-valued b-metric spaces. Using the condition (jj), from (10) we deduce

$$D_b(G_T Y, G_T V) \leq \|\alpha\| D_b(G_T Y, Y) + \|\beta\| D_b(G_T V, V) \quad \text{for all } Y, V \in X \times X.$$

Since $(X \times X, D_b, \|b\|)$ is a complete b-metric space the result follows by Lemma 4 with $a_1 = \|\alpha\|$, $a_2 = \|\beta\|$ and $a_3 = a_4 = a_5 = 0$. □

If in the previous theorem we assume $b = I$, then we deduce the following corollary.

Corollary 12. *Let $(X, \mathscr{A}, d)$ be a complete C^*-algebra-valued metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following condition:*

$$d\left(F(z,w), F(u,v)\right) + d\left(F(w,z), F(v,u)\right)$$

$$\preceq \alpha\left(d\left(F(z,w), z\right) + d\left(F(w,z), w\right)\right) + \beta\left(d\left(F(u,v), u\right) + d\left(F(v,u), v\right)\right) \tag{11}$$

for any $z,w,u,v \in X$, where $\alpha, \beta \in \mathscr{A}$ with $\|\alpha\| + \|\beta\| < 1$. Then F has a unique coupled fixed point.

Remark 13. *Theorem 11 generalizes Theorem 2.3 of [5] and Corollary 12 generalizes Theorem 2.2 of [19].*

The following is a Chatterjea type result of coupled fixed point.

Theorem 14. *Let* $(X,\mathscr{A},d_b)$ *be a complete* C^**-algebra-valued b-metric space. Suppose that the mapping* $T : X\times X \to X$ *satisfies the following condition*

$$d_b\left(T\left(z,w\right),T\left(u,v\right)\right)+d_b\left(T\left(w,z\right),T\left(v,u\right)\right)$$

$$\preceq \alpha\left(d_b\left(T\left(z,w\right),u\right)+d_b\left(T\left(w,z\right),v\right)\right)+\beta\left(d_b\left(T\left(u,v\right),z\right)+d_b\left(T\left(v,u\right),w\right)\right) \tag{12}$$

for any $z,w,u,v\in X$, *where* $\alpha,\beta\in\mathscr{A}$ *with* $\|b\|(1+\|b\|)\|\alpha\|+\|\beta\|)<2$. *Then* T *has a unique coupled fixed point.*

Proof. We have to verify that the mapping $G_T : X\times X \to X\times X$ has a unique fixed point. Now, we see that (12) implies

$$d_{b^+}(G_TY,G_TV))\preceq \alpha d_{b+}(G_TY,V)+\beta d_{b^+}(G_TV,Y), \tag{13}$$

for all $Y,V\in X\times X$, that is., G_T is a Chatterjea type contractive mapping in the framework of C^*-algebra-valued b-metric spaces. Using the condition (jj), from (13) we deduce

$$D_b(G_TY,G_TV)\leq \|\alpha\|D_b(G_TY,V)+\|\beta\|D_b(G_TV,Y) \quad \text{for all } Y,V\in X\times X.$$

Since $(X\times X,D_b,\|b\|)$ is a complete b-metric space the result follows by Lemma 4 with $a_4=\|\alpha\|$, $a_5=\|\beta\|$ and $a_1=a_2=a_3=0$. □

If in the previous theorem we assume $b=I$, then we deduce the following corollary.

Corollary 15. *Let* $(X,\mathscr{A},d)$ *be a complete* C^**-algebra-valued metric space. Suppose that the mapping* $F : X\times X\to X$ *satisfies the following condition:*

$$d\left(F\left(z,w\right),F\left(u,v\right)\right)+d\left(F\left(w,z\right),F\left(v,u\right)\right)$$

$$\preceq \alpha\left(d\left(F\left(z,w\right),u\right)+d\left(F\left(w,z\right),v\right)\right)+\beta\left(d\left(F\left(u,v\right),z\right)+d\left(F\left(v,u\right),w\right)\right), \tag{14}$$

for any $z,w,u,v\in X$, *where* $\alpha,\beta\in\mathscr{A}$ *with* $\|\alpha\|+\|\beta\|<1$. *Then* F *has a unique coupled fixed point.*

Remark 16. *Theorem 14 is a generalization of Theorem 2.2 of [5] and Corollary 15 is a generalization of Theorem 2.3 of [19].*

Remark 17. *The proofs in [2] and [19] are correct but with stronger assumptions. While, our proofs are much shorter and nicer with weaker assumptions, that is.,* $\alpha,\beta\in\mathscr{A}$ *instead* $\alpha,\beta\in\mathscr{A}'_+=\{\gamma\in\mathscr{A}_+ : \gamma\delta=\delta\gamma \text{ for all } \delta\in\mathscr{A}_+\}$, *for example. It is also worth to noticing that examples as well as applications given in [2] and [19] support in the fact our approach.*

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