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Some New Results in the Framework of S_b- Metric Spaces

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Abstract: In this paper, we consider S_b- metric space as a generalization of metric space. Also we obtained some common fixed point results using C–class function with refinement inequality in S_b–metric spaces. Further, we present one example, which shows that obtained results are proper generalization of the results in literature.

Keywords: Common fixed point, C -class function, Refinement inequality, S_b–metric space.

1 Introduction and Preliminaries

Banach contraction principle [4] is a most celebrated result due to its several applications in various branches of sciences particularly topology, functional and nonlinear analysis. The generalization of Banach contraction principle in different branches of sciences remain an active area of research. Fixed point theory has many applications in various fields, especially computer sciences, dynamic analysis, image processing and optimizations. Due to various applications of fixed point theory, many researchers studied and generalized the Banach contraction principle, see ([1]-[17],[19]-[12]). Also, the generalization of metric spaces in various structure has attracted attention of scientists due to the development and generalization of fixed point theory in metric spaces. Particularly, Bakhtin [3] generalized the concept of metric space by introducing $b-$metric space, many researchers studied fixed point point results for various mappings satisfying certain conditions in $b-$metric space. Sedghi and Shobe [17] proved a common fixed point of four mappings in complete metric spaces. Later, Abbas et al. in [1] proved a common fixed points of four mappings satisfying a generalized weak contractive condition in the partially ordered metric spaces. Mean while Roshan et al. [13] proved a common fixed point result of four maps in b-metric spaces.

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Later, Sedghi et al. [15] that is., Sedghi et al. [18] generalized the concept of metric space to $S-$metric space, that is., to S_b-metric space. For more details on S_b-metric spaces, also see ([9], [12], [18]). The aim of this paper is to further consider S_b-metric space and to present some common fixed point results for four mappings satisfying generalized contractive condition in a S_b-metric space, where the S_b-metric is not necessary continuous.

First, we recall some notions, results and examples, which are required in sequel.

Definition 1. *[15] Let X be a nonempty set. A $S-$metric on X is a function $S : X^3 \to [0,\infty)$ that satisfies the following conditions for all $x,y,z,a \in X$.*

(S2) $S(x,y,z) = 0$ *if and only* $x = y = z$,

(S3) $S(x,y,z) \leq S(x,x,a)+S(y,y,a)+S(z,z,a)$ *for all* $x,y,z,a \in X$.

The pair (X,S) is called a $S-$metric space.

Example 1. *[15] Let $X = \mathbb{R}^2$ and d be an ordinary metric on X, and $S(x,y,z) = d(x,y) + d(x,z)+d(y,z)$ for all $x,y,z \in \mathbb{R}^2$, is a S-metric on X.*

Lemma 1. *[14] Let (X,S) be a S-metric space, then $S(x,x,y) = S(y,y,x)$ for all $x,y \in X$.*

Definition 2. *[16] Let (X,S) be a $S-$metric space. For $r > 0$ and $x \in X$, define an open ball $B_S(x,r)$ and closed ball $B_S\,[x,r]$ with center x and radius r as follows respectively:*

$$\begin{aligned} B_s(x,r) &= \{y \in X : S(y,y,x) < r\}, \\ B_s[x,r] &= \{y \in X : S(y,y,x) \leq r\}. \end{aligned}$$

Definition 3. *[16] Let (X,S) be a $S-$metric space and $A \subseteq X$.*

(1) If for every $x \in X$ there exists $r > 0$ such that $B_s(x,r) \subseteq A$, then the subset A is called open subset of X.

(2) For some $A \subseteq X$ is said to be $S-$bounded if there exists $r > 0$ such that $S(x,x,y) < r$ for all $x,y \in A$.

(3) A sequence $\{x_n\}$ in X is S-convergent and S-converges to x if and only if $S(x_n,x_n,x) \to 0$ as $n \to \infty$. That is for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $S(x_n,x_n,x) < \varepsilon$ and denote $n \longrightarrow \infty \lim x_n = x$.

(4) A sequence $\{x_n\}$ in X is called a S-Cauchy sequence if for each $\varepsilon > 0$, there exists for some $n_0 \in \mathbb{N}$ such that for each $n,m \geq n_0$, $S(x_n,x_n,x_m) < \varepsilon$.

(5) *The S-metric space (X,S) is said to be S-complete if every S-Cauchy sequence is convergent.*

Lemma 2. *[16] Let (X,S) be a $S-$metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$in X such that $x_n \to x$ and $y_n = y$ then $S(x_n,x_n,y_n) \to S(x,x,y)$ as $n \to \infty$.*

The notion of S_b-metric space introduced in [18] as a generalization of $S-$ metric space due to Czerwik [5] and Bakhtin [3] in which the triangular inequality has been replaced by a weaker condition, as follows.

Definition 4. *[18] Let X be a nonempty set and $b \geq 1$ be any real number. Suppose that a mapping $S_b : X^3 \to [0,\infty)$ satisfies:*

(S_b2) $S_b(x,y,z) = 0$ *if and only if* $x = y = z$,

(S_b3) $S_b(x,y,z) \leq b(S_b(x,x,a)+S_b(y,y,a)+S_b(z,z,a))$ *for all* $x,y,z,a \in X$.
Then S_b is called a S_b-metric and the pair (X,S_b) is called a S_b-metric space.

Remark 1. *It should be noted that, the class of S_b-metric spaces is effectively larger than that of $S-$metric spaces. Indeed each S-metric space is a S_b-metric space with $b = 1$.*

Following example shows that a S_b-metric on X need not be a $S-$metric on X.

Example 2. *[18] Let (X,S) be a $S-$metric space and $S_b(x,y,z) = (S(x,y,z))^p$, where $p > 1$ is a real number. Note that S_b is a S_b-metric with $b = 2^{2(p-1)}$, is a S_b-metric but every S_b-metric need not to be $S-$metric.*

Now, we present some definitions and propositions in S_b-metric space.

Definition 5. *[18] Let (X,S_b) be a S_b-metric space. Then, for any $x \in X$, $r > 0$, define open $B_{S_b}(x,r)$ and closed ball $B_{S_b}[x,r]$ with center x, radius r as in the case of S-metric spaces (see Definition 1.4.).*

Example 3. *[18] Let (X,S_b) be a S_b-metric space on $\mathbb{R}$ with $b = 2^{2(2-1)} = 4$, where*

$$S_b(x,y,z) = (|y+z-2x| + |y-z|)^2$$

for all $x,y,z \in \mathbb{R}$. Thus

$$B_{S_b}(1,2) = \{y \in \mathbb{R} : S_b(y,y,1) < 2\},$$

and

$$B_{S_b}[1,2] = \{y \in \mathbb{R} : S_b(y,y,1) \leq 2\}.$$

Lemma 3. *[18] Let (X,S_b) be a S_b-metric space, then $S_b(x,x,y) \leq bS_b(y,y,x)$ and $S_b(y,y,x) \leq bS_b(x,x,y)$, for all x,y in X.*

Lemma 4. *[18] Let (X,S_b) be a S_b-metric space, then $S_b(x,x,z) \leq 2bS(x,x,z)+b^2S_b(y,y,z)$, for all x,y,z in X.*

Definition 6. *[18] Let (X,S_b) be a S_b-metric space. A sequence $\{x_n\}$ in X is:*

1. *S_b-Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S_b(x_n,x_n,x_m) < \varepsilon$ for all $m,n \geq n_0$.*
2. *S_b-convergent and converges to $x \in X$ if for each $\varepsilon > 0$, there exists a positive integer n_0 such that $S_b(x_n,x_n,x) < \varepsilon$ or $S_b(x,x,x_n) < \varepsilon$, for all $n \geq n_0$ and we denote it $\lim_{n\longrightarrow\infty} x_n = x$.*

Definition 7. *[18] A S_b-metric space (X,S_b) is S_b-complete if every S_b-Cauchy sequence is S_b-convergent in X.*

Definition 8. *[18] Let (X,S_b) and (X',S'_b) are S_b-metric spaces and let $f:(X,S_b) \to (X',S'_b)$ be a function. Then f is said to be continuous at a point $x \in X$ if and only if*

$$S_b(x_n,x_n,x) \to 0 \text{ implies } S'_b(f(x_n),f(x_n),f(x)) \to 0$$

for every sequence x_n in X which converges to $x \in X$.
A function f is continuous at X if and only if it is continuous at all $x \in X$.

Definition 9. *[18] Let (X,S_b) be a S_b-metric space. A pair $\{f,g\}$ is said to be S_b-compatible if and only if $S_b(fgx_n,fgx_n,gfx_n) \to 0$, whenever $\{x_n\}$ is a sequence in X such that $fx_n, gx_n \to y$, for some $y \in X$.*

Lemma 5. *[18] Let (X,S_b) be a S_b-metric space with $b \geq 1$ and suppose that $\{x_n\}$ is S_b-convergent and converges to x then*

$$\frac{1}{b^2}S_b(x,x,y) \leq \liminf_{n\to\infty} S_b(x_n,x_n,y) \leq \limsup_{n\to\infty} S_b(x_n,x_n,y) \leq b^2S_b(x,x,y).$$

In particular, if $x = y$, then $S_b(x_n,x_n,y) \to 0$ as $n \to \infty$.

Lemma 6. *[18] Let (X,S_b) be a S_b-metric space. If there exist sequences $\{x_n\},\{y_n\}$ in X such that $S_b(x_n,x_n,y_n) \to 0$ and $x_n \to x$ for some $x \in X$, then $y_n \to x$ as $n \to \infty$.*

In 2014, Ansari [2] introduced the concept of a C-class functions which cover a large class of contractive conditions.

Definition 10. *[2] A continuous function $F:[0,\infty)^2 \to \mathbb{R}$ is called C-class function if for any $s,t \in [0,\infty)$, the following conditions hold:*

1) $F(s,t) \leq s$;

2) $F(s,t) = s$ *implies that either* $s = 0$ *or* $t = 0$.

Here $\mathscr{C}$ denote the class of all *C-class* functions.

Let Φ_u denote the class of the functions $\varphi : [0,\infty) \to [0,\infty)$ which satisfy the following conditions:

(a) φ is continuous,

(b) $\varphi(t) > 0$ for all $t > 0$ and $\varphi(0) \geq 0$.

Definition 11. *([8])A function* $\psi : [0,\infty) \to [0,\infty)$ *is called an altering distance function if the following properties are satisfied:*

i) ψ is non-decreasing and continuous,

ii) $\psi(t) = 0$ if and only if $t = 0$.

Ψ denote the class of altering distance functions.

Definition 12. *A tripled* (ψ,φ,F) *is said to be monotone if for any* $x,y \in [0,\infty)$ $x \leqslant y$ *implies* $F(\psi(x),\varphi(x)) \leqslant F(\psi(y),\varphi(y))$ *where* $\psi \in \Psi$, $\varphi \in \Phi_u$ *and* $F \in \mathscr{C}$.

Example 4. *Let* $F(s,t) = s - t$, $\phi(x) = \sqrt{x}, \psi(x) = \sqrt{x}$ *if* $x \in [0,1]$, *that is* $\psi(x) = x^2$ *if* $x > 1$. *Then* (ψ,ϕ,F) *is monotone.*

Example 5. *Let* $F(s,t) = s - t$, $\phi(x) = x^2, \psi(x) = \sqrt{x}$ *if* $x \in [0,1]$, *that is* $\psi(x) = x^2$ *if* $x > 1$. *Then* (ψ,ϕ,F) *is not monotone.*

2 Main Results

In our first result, we discuss common fixed point for four mappings.

Theorem 1. *Suppose that* f, g, h *and* T *are self mappings on complete* S_b*-metric space* (X,S_b) *such that* $f(X) \subseteq T(X)$, $g(X) \subseteq h(X)$. *If*

$$\psi(b^4 S_b(fx,fx,gy))) \leq F[\psi(L(x,y)),\varphi(L(x,y))]. \tag{1}$$

Where

$$L(x,y) = \max\{S_b(hx,hx,Ty), S_b(fx,fx,hx), S_b(gy,gy,Ty), \frac{1}{2}(S_b(hx,hx,gy) + S_b(fx,fx,Ty))\},$$

holds for all $x,y \in X$ *with* $F \in C, \psi \in \Psi, \varphi \in \Phi_u$ *where* (ψ,ϕ,F) *is monotone and* $\frac{2b}{2b^2-1} < 1$ $(\approxeq b > \frac{1+\sqrt{3}}{2})$, *then* f, g, h *and* T *have a unique common fixed point in* X *provided that* h *and* T *are continuous, pairs* $\{f,h\}$ *and* $\{g,T\}$ *are* S_b-*compatible.*

Proof. Let $x_0 \in X$, as $f(X) \subseteq T(X)$, then there exists some $x_1 \in X$ such that $fx_0 = Tx_1$. Since $gx_1 \in h(X)$, we can choose $x_2 \in X$ such that $gx_1 = hx_2$. In general, x_{2n+1} and x_{2n+2} are chosen in X such that $fx_{2n} = Tx_{2n+1}$ and $gx_{2n+1} = hx_{2n+2}$. Define a sequence y_n in X such that $y_{2n} = fx_{2n} = Tx_{2n+1}$ and $y_{2n+1} = gx_{2n+1} = hx_{2n+2}$ for all $n \geq 0$. Now, we have to show that $\{y_n\}$ is a Cauchy sequence. Consider

$$
\begin{aligned}
& \psi(b^4 S_b(y_{2n}, y_{2n}, y_{2n+1})) = \psi(b^4 S_b(fx_{2n}, fx_{2n}, gx_{2n+1})) \\
\leq\ & F(\psi(\max\{S_b(hx_{2n}, hx_{2n}, Tx_{2n+1}), S_b(fx_{2n}, fx_{2n}, hx_{2n}), S_b(gx_{2n+1}, gx_{2n+1}, Tx_{2n+1}), \\
& \frac{1}{2}(S_b(hx_{2n}, hx_{2n}, gx_{2n+1}) + S_b(fx_{2n}, fx_{2n}, Tx_{2n+1}))\}), \\
& \varphi(\max\{S_b(hx_{2n}, hx_{2n}, Tx_{2n+1}), S_b(fx_{2n}, fx_{2n}, hx_{2n}), S_b(gx_{2n+1}, gx_{2n+1}, Tx_{2n+1}), \\
& \frac{1}{2}(S_b(hx_{2n}, hx_{2n}, gx_{2n+1}) + S_b(fx_{2n}, fx_{2n}, Tx_{2n+1}))\})) \\
=\ & F(\psi(\max\{S_b(y_{2n-1}, y_{2n-1}, y_{2n}), S_b(y_{2n}, y_{2n}, y_{2n-1}), S_b(y_{2n+1}, y_{2n+1}, y_{2n}), \\
& \frac{1}{2}(S_b(y_{2n-1}, y_{2n-1}, y_{2n+1}) + S_b(y_{2n}, y_{2n}, y_{2n}))\}), \\
& \varphi(\max\{S_b(y_{2n-1}, y_{2n-1}, y_{2n}), S_b(y_{2n}, y_{2n}, y_{2n-1}), S_b(y_{2n+1}, y_{2n+1}, y_{2n}), \\
& \frac{1}{2}(S_b(y_{2n-1}, y_{2n-1}, y_{2n+1}) + S_b(y_{2n}, y_{2n}, y_{2n}))\}), \\
\leq\ & F(\psi(\max\{S_b(y_{2n-1}, y_{2n-1}, y_{2n}), bS_b(y_{2n-1}, y_{2n-1}, y_{2n}), \\
& S_b(y_{2n+1}, y_{2n+1}, y_{2n}), \frac{S_b(y_{2n-1}, y_{2n-1}, y_{2n+1})}{2}\}), \\
& \varphi(\max\{S_b(y_{2n-1}, y_{2n-1}, y_{2n}), bS_b(y_{2n-1}, y_{2n-1}, y_{2n}), \\
& S_b(y_{2n+1}, y_{2n+1}, y_{2n}), \frac{S_b(y_{2n-1}, y_{2n-1}, y_{2n+1})}{2}\})) \\
\leq\ & F(\psi(\max\{S_b(y_{2n-1}, y_{2n-1}, y_{2n}), bS_b(y_{2n-1}, y_{2n-1}, y_{2n}), S_b(y_{2n+1}, y_{2n+1}, y_{2n}), \\
& \frac{b}{2}(S_b(y_{2n-1}, y_{2n-1}, y_{2n}) + S_b(y_{2n-1}, y_{2n-1}, y_{2n}) + S_b(y_{2n+1}, y_{2n+1}, y_{2n}))\}), \\
& \varphi(\max\{S_b(y_{2n-1}, y_{2n-1}, y_{2n}), bS_b(y_{2n-1}, y_{2n-1}, y_{2n}), S_b(y_{2n+1}, y_{2n+1}, y_{2n}), \\
& \frac{b}{2}(S_b(y_{2n-1}, y_{2n-1}, y_{2n}) + S_b(y_{2n-1}, y_{2n-1}, y_{2n}) + S_b(y_{2n+1}, y_{2n+1}, y_{2n}))\})) \\
\leq\ & F(\psi(\max\{S_b(y_{2n-1}, y_{2n-1}, y_{2n}), bS_b(y_{2n-1}, y_{2n-1}, y_{2n}), bS_b(y_{2n}, y_{2n}, y_{2n+1}), \\
& \frac{b}{2}(2S_b(y_{2n-1}, y_{2n-1}, y_{2n}) + bS_b(y_{2n+1}, y_{2n+1}, y_{2n}))\}), \\
& \varphi(\max\{S_b(y_{2n-1}, y_{2n-1}, y_{2n}), bS_b(y_{2n-1}, y_{2n-1}, y_{2n}), bS_b(y_{2n}, y_{2n}, y_{2n+1}), \\
& \frac{b}{2}(2S_b(y_{2n-1}, y_{2n-1}, y_{2n}) + bS_b(y_{2n+1}, y_{2n+1}, y_{2n}))\}).
\end{aligned}
$$

Since

$$S_b(y_{2n-1},y_{2n-1},y_{2n}) \leq bS_b(y_{2n-1},y_{2n-1},y_{2n}) \leq bS_b(y_{2n-1},y_{2n-1},y_{2n}) + \frac{b^2}{2}S_b(y_{2n},y_{2n},y_{2n+1}),$$

we have

$$\begin{aligned}
& \psi(b^4 S_b(y_{2n},y_{2n},y_{2n+1})) \\
\leq\ & F(\psi(\max\{bS_b(y_{2n},y_{2n},y_{2n+1}), bS_b(y_{2n-1},y_{2n-1},y_{2n}) + \frac{b^2}{2}S_b(y_{2n},y_{2n},y_{2n+1})\}) \\
& \varphi(\max\{bS_b(y_{2n},y_{2n},y_{2n+1}), bS_b(y_{2n-1},y_{2n-1},y_{2n}) + \frac{b^2}{2}S_b(y_{2n},y_{2n},y_{2n+1})\})) \\
\leq\ & \psi(\max\{bS_b(y_{2n},y_{2n},y_{2n+1}), bS_b(y_{2n-1},y_{2n-1},y_{2n}) + \frac{b^2}{2}S_b(y_{2n},y_{2n},y_{2n+1})\}).
\end{aligned}$$

If

$$\max\{bS_b(y_{2n},y_{2n},y_{2n+1}), bS_b(y_{2n-1},y_{2n-1},y_{2n}) + \frac{b^2}{2}S_b(y_{2n},y_{2n},y_{2n+1})\} = bS_b(y_{2n},y_{2n},y_{2n+1}),$$

then above inequality becomes

$$\begin{aligned}
\psi(bS_b(y_{2n},y_{2n},y_{2n+1})) &\leq \psi(b^4 S_b(y_{2n},y_{2n},y_{2n+1})) \\
&\leq F[\psi(bS_b(y_{2n},y_{2n},y_{2n+1})), \varphi(bS_b(y_{2n},y_{2n},y_{2n+1}))],
\end{aligned}$$

implies either

$$\psi(bS_b(y_{2n},y_{2n},y_{2n+1})) = 0 \quad \text{or } \varphi(bS_b(y_{2n},y_{2n},y_{2n+1})) = 0,$$

therefore $S_b(y_{2n},y_{2n},y_{2n+1}) = 0$, a contradiction. So

$$\begin{aligned}
& \max\{bS_b(y_{2n},y_{2n},y_{2n+1}), bS_b(y_{2n-1},y_{2n-1},y_{2n}) + \frac{b^2}{2}S_b(y_{2n},y_{2n},y_{2n+1})\} \\
=\ & bS_b(y_{2n-1},y_{2n-1},y_{2n}) + \frac{b^2}{2}S_b(y_{2n},y_{2n},y_{2n+1}),
\end{aligned}$$

then, we have

$$\begin{aligned}
& \psi(b^4 S_b(y_{2n},y_{2n},y_{2n+1})) \leq \\
\leq\ & F(\psi(bS_b(y_{2n-1},y_{2n-1},y_{2n}) + \frac{b^2}{2}S_b(y_{2n},y_{2n},y_{2n+1})), \varphi(bS_b(y_{2n-1},y_{2n-1},y_{2n}) + \\
& + \frac{b^2}{2}S_b(y_{2n},y_{2n},y_{2n+1}))) \leq \\
\leq\ & \psi(bS_b(y_{2n-1},y_{2n-1},y_{2n}) + \frac{b^2}{2}S_b(y_{2n},y_{2n},y_{2n+1})),
\end{aligned}$$

which further implies

$$b^4 S_b(y_{2n}, y_{2n}, y_{2n+1}) \leq b S_b(y_{2n-1}, y_{2n-1}, y_{2n}) + \frac{b^2}{2} S_b(y_{2n}, y_{2n}, y_{2n+1})$$

that is

$$S_b(y_{2n}, y_{2n}, y_{2n+1}) \leq \frac{2}{b(2b^2-1)} S_b(y_{2n-1}, y_{2n-1}, y_{2n}).$$

Let us consider, $\lambda = \frac{2}{b(2b^2-1)}$. Since $b > \frac{1+\sqrt{3}}{2}$, we have $0 < \lambda < 1$. Hence

$$\begin{aligned} & S_b(y_{2n}, y_{2n}, y_{2n+1}) \leq \lambda S_b(y_{2n-1}, y_{2n-1}, y_{2n}) \leq \lambda^2 S_b(y_{2n-2}, y_{2n-2}, y_{2n-1}) \leq \cdots \leq \\ \leq \ & \lambda^n S_b(y_0, y_0, y_1), \end{aligned}$$

for all $n \geq 2$, we have

$$S_b(y_{n-1}, y_{n-1}, y_n) \leq ... \leq \lambda^{n-1} S_b(y_0, y_0, y_1). \tag{2}$$

Using Lemma 3, $(S_b 3)$ and (2) for all $n > m$, we have

$$\begin{aligned} S_b(y_m, y_m, y_n) &\leq b(2S_b(y_m, y_m, y_{m+1}) + S_b(y_n, y_n, y_{m+1})) \\ &\leq 2bS_b(y_m, y_m, y_{m+1}) + b^2 S_b(y_{m+1}, y_{m+1}, y_n) \\ &\leq 2bS_b(y_m, y_m, y_{m+1}) + 2b^3 S_b(y_{m+1}, y_{m+1}, y_{m+2}) + b^4 S_b(y_{m+2}, y_{m+2}, y_n) \\ &\vdots \\ &\leq 2b(S_b(y_m, y_m, y_{m+1}) + b^2 S_b(y_{m+1}, y_{m+1}, y_{m+2}) + \cdots + \\ &\quad + b^{2(n-m-1)} S_b(y_{n-1}, y_{n-1}, y_n)) \\ &\leq 2b(\lambda^m + b^2 \lambda^{m+1} + \cdots + b^{2(n-m-1)} \lambda^{n-1}) S_b(y_0, y_0, y_1) \\ &\leq 2bS_b(y_0, y_0, y_1)(\lambda^m + b^2 \lambda^{m+1} + \ldots) \\ &\leq \frac{2b\lambda^m}{1 - b^2\lambda} S_b(y_0, y_0, y_1). \end{aligned}$$

On taking limit as $m, n \to \infty$, we have $S_b(y_m, y_m, y_n) \to 0$ as $b^2\lambda < 1$. Therefore $\{y_n\}$ is a S_b-Cauchy sequence. Since X is a S_b-complete metric space, there exist some y in X such that

$$\lim_{n \longrightarrow \infty} f x_{2n} = \lim_{n \longrightarrow \infty} T x_{2n+1} = \lim_{n \longrightarrow \infty} g x_{2n+1} = \lim_{n \longrightarrow \infty} h x_{2n+2} = y.$$

Now, we have to show that y is a common fixed point of f, g, h and T. Since h is continuous, therefore

$$\lim_{n \longrightarrow \infty} h^2 x_{2n+2} = hy \qquad \text{and} \qquad \lim_{n \longrightarrow \infty} hf x_{2n} = hy.$$

Since a pair $\{f,h\}$ is compatible, $\lim_{n\longrightarrow\infty} S_b(fhx_{2n},fhx_{2n},hfx_{2n})=0$. So by Lemma 6, we have

$$\lim_{n\longrightarrow\infty} fhx_{2n}=hy.$$

Putting $x=hx_{2n}$ and $y=x_{2n+1}$ in (1), we obtain

$$\begin{aligned}
& \psi(b^4S_b(fhx_{2n},fhx_{2n},gx_{2n+1}) \qquad (3)\\
\leq\ & F(\psi(\max\{S_b(h^2x_{2n},h^2x_{2n},Tx_{2n+1}),S_b(fhx_{2n},fhx_{2n},h^2x_{2n}),\\
& S_b(gx_{2n+1},gx_{2n+1},Tx_{2n+1}),\frac{1}{2}(S_b(h^2x_{2n},h^2x_{2n},gx_{2n+1})+S_b(fhx_{2n},fhx_{2n},Tx_{2n+1}))\}),\\
& \varphi(\max\{S_b(h^2x_{2n},h^2x_{2n},Tx_{2n+1}),S_b(fhx_{2n},fhx_{2n},h^2x_{2n}),S_b(gx_{2n+1},gx_{2n+1},Tx_{2n+1}),\\
& \frac{1}{2}(S_b(h^2x_{2n},h^2x_{2n},gx_{2n+1})+S_b(fhx_{2n},fhx_{2n},Tx_{2n+1}))\})).
\end{aligned}$$

Taking upper limit as $n\to\infty$ in (3) and using Lemma 5, we have

$$\begin{aligned}
& \psi(\frac{b^4S_b(hy,hy,y)}{b^2})\\
\leq\ & \psi(\limsup_{n\longrightarrow\infty} b^4S_b(fhx_{2n},fhx_{2n},gx_{2n+1}))\\
\leq\ & F(\psi(\max\{\limsup_{n\longrightarrow\infty} S_b(h^2x_{2n},h^2x_{2n},Tx_{2n+1}),\limsup_{n\longrightarrow\infty} S_b(fhx_{2n},fhx_{2n},h^2x_{2n}),\\
& \limsup_{n\longrightarrow\infty} S_b(gx_{2n+1},gx_{2n+1},Tx_{2n+1}),\frac{1}{2}(\limsup_{n\longrightarrow\infty} S_b(h^2x_{2n},h^2x_{2n},gx_{2n+1})+\\
& \limsup_{n\longrightarrow\infty} S_b(fhx_{2n},fhx_{2n},Tx_{2n+1}))\}),\varphi(\max\{\limsup_{n\longrightarrow\infty} S_b(h^2x_{2n},h^2x_{2n},Tx_{2n+1}),\\
& \limsup_{n\longrightarrow\infty} S_b(fhx_{2n},fhx_{2n},h^2x_{2n}),\limsup_{n\longrightarrow\infty} S_b(gx_{2n+1},gx_{2n+1},Tx_{2n+1}),\\
& \frac{1}{2}(\limsup_{n\longrightarrow\infty} S_b(h^2x_{2n},h^2x_{2n},gx_{2n+1})+\limsup_{n\longrightarrow\infty} S_b(fhx_{2n},fhx_{2n},Tx_{2n+1}))\}))\\
\leq\ & F(\psi(\max\{b^2S_b(hy,hy,y),0,0,\frac{b^2}{2}(S_b(hy,hy,y)+S_b(hy,hy,y))\}),\\
& \varphi(\max\{b^2S_b(hy,hy,y),0,0,\frac{b^2}{2}(S_b(hy,hy,y)+S_b(hy,hy,y))\})\\
=\ & F(\psi(b^2S_b(hy,hy,y)),\varphi(b^2S_b(hy,hy,y))).
\end{aligned}$$

Either

$$\psi(b^2S_b(hy,hy,y))=0 \text{ or } \varphi(b^2S_b(hy,hy,y))=0,$$

therefore

$$b^2S_b(hy,hy,y)=0.$$

Consequently, $hy=y$. Since T is continuous mapping, so we obtain

$$\lim_{n\longrightarrow\infty} T^2x_{2n+1}=Ty \text{ and } \lim_{n\longrightarrow\infty} Tgx_{2n+1}=Ty.$$

Since g and T are compatible, $\lim_{n\longrightarrow\infty} S_b(gTx_n,gTx_n,Tgx_n)=0$. By Lemma 6, we have

$$\lim_{n\longrightarrow\infty} gTx_{2n}=Ty.$$

Putting $x=x_{2n}$ and $y=Tx_{2n+1}$ in (1), we obtain

$$\begin{aligned}
&\psi(b^4S_b(fx_{2n},fx_{2n},gTx_{2n+1})) \qquad (4)\\
\leq\ & F(\psi(\max\{S_b(hx_{2n},hx_{2n},T^2x_{2n+1}),S_b(fx_{2n},fx_{2n},hx_{2n}),S_b(gTx_{2n+1},gTx_{2n+1},T^2x_{2n+1}),\\
&\frac{1}{2}(S_b(hx_{2n},hx_{2n},gTx_{2n+1})+S_b(fx_{2n},fx_{2n},T^2x_{2n+1}))\}),\\
&\varphi(\max\{S_b(hx_{2n},hx_{2n},T^2x_{2n+1}),S_b(fx_{2n},fx_{2n},hx_{2n}),S_b(gTx_{2n+1},gTx_{2n+1},T^2x_{2n+1}),\\
&\frac{1}{2}(S_b(hx_{2n},hx_{2n},gTx_{2n+1})+S_b(fx_{2n},fx_{2n},T^2x_{2n+1}))\}).
\end{aligned}$$

Taking limit as $n\to\infty$ in (4) and using Lemma 5, we obtain

$$\begin{aligned}
\psi(\frac{b^4S_b(y,y,Ty)}{b^2}) &\leq \psi(\limsup_{n\longrightarrow\infty} b^4S_b(fx_{2n},fx_{2n},gTx_{2n+1}))\\
&\leq F(\psi(\max\{b^2(S_b(y,y,Ty),0,0,\frac{b^2}{2}S_b(y,y,Ty)+S_b(y,y,Ty))\}),\\
&\quad \varphi(\max\{b^2(S_b(y,y,Ty),0,0,\frac{b^2}{2}S_b(y,y,Ty)+S_b(y,y,Ty))\}))\\
&= F(\psi(b^2S_b(y,y,Ty)),\varphi(b^2S_b(y,y,Ty))).
\end{aligned}$$

Implies that either

$$\psi(b^2S_b(y,y,Ty))=0 \text{ or } \varphi(b^2S_b(y,y,Ty))=0,$$

therefore

$$b^2S_b(y,y,Ty)=0,$$

which implies that $Ty=y$. Apply condition (1) to obtain

$$\begin{aligned}
&\psi(b^4S_b(fy,fy,gx_{2n+1}))\leq F(\psi(\max\{S_b(hy,hy,Tx_{2n+1}),S_b(fy,fy,hy), \qquad (5)\\
&S_b(gx_{2n+1},gx_{2n+1},Tx_{2n+1}),\\
&\frac{1}{2}(S_b(hy,hy,gx_{2n+1})+S_b(fy,fy,Tx_{2n+1}))\}),\\
&\varphi(\max\{S_b(hy,hy,Tx_{2n+1}),S_b(fy,fy,hy),S_b(gx_{2n+1},gx_{2n+1},Tx_{2n+1}),\\
&\frac{1}{2}(S_b(hy,hy,gx_{2n+1})+S_b(fy,fy,Tx_{2n+1}))\})).
\end{aligned}$$

By applying limit $n \to \infty$ in (6), and using $hy = Ty = y$, we have

$$\begin{aligned}
\psi(b^2 S_b(fy, fy, y)) \quad &\leq \quad F(\psi(\max\{b^2 S_b(hy, hy, y), b^2 S_b(fy, fy, hy), b^2 S_b(y, y, y), \\
&\quad \frac{b^2}{2}(S_b(hy, hy, y) + S_b(fy, fy, y))), \\
&\quad \varphi(\max\{b^2 S_b(hy, hy, y), b^2 S_b(fy, fy, hy), b^2 S_b(y, y, y), \\
&\quad \frac{b^2}{2}(S_b(hy, hy, y) + S_b(fy, fy, y))), \\
&= \quad F(\psi(b^2 S_b(fy, fy, y)), \varphi(b^2 S_b(fy, fy, y))).
\end{aligned}$$

Again either

$$\psi(b^2 S_b(fy, fy, y)) = 0 \text{ or } \varphi(b^2 S_b(fy, fy, y)) = 0,$$

which implies that $S_b(fy, fy, y) = 0$ and $fy = y$ as $0 < q < 1$.

Finally, from condition (1) and the fact $hy = Ty = fy = y$, we have

$$\begin{aligned}
\psi(b^4 S_b(y, y, gy)) \quad &= \quad \psi(b^4 S_b(fy, fy, gy)) \\
&\leq \quad F(\psi(\max\{S_b(hy, hy, Ty), S_b(fy, fy, hy), S_b(gy, gy, Ty), \\
&\quad \frac{1}{2}(S_b(hy, hy, gy) + S_b(fy, fy, Ty))), \\
&\quad \varphi(\max\{S_b(hy, hy, Ty), S_b(fy, fy, hy), S_b(gy, gy, Ty), \\
&\quad \frac{1}{2}(S_b(hy, hy, gy) + S_b(fy, fy, Ty)))) \\
&\leq \quad F(\psi(bS_b(y, y, gy)), \varphi(bS_b(y, y, gy))),
\end{aligned}$$

We have

$$\psi(b^2 S_b(y, y, gy)) = 0 \text{ or } \varphi(b^2 S_b(y, y, gy)) = 0,$$

which implies that $S_b(y, y, gy) = 0$ and $gy = y$. Hence $hy = Ty = fy = gy = y$.

To show the uniqueness: On contrary suppose that if there exists another common fixed point x in X for f, g, h and T, then

$$\begin{aligned}
& \psi(b^4 S_b(x,x,y)) = \psi(b^4 S_b(fx,fx,gy)) \\
\leq\ & F(\psi(\max\{S_b(hx,hx,Ty), S_b(fx,fx,hx), S_b(gy,gy,Ty), \\
& \frac{1}{2}(S_b(hx,hx,gy)+S_b(fx,fx,Ty))\}), \\
& \varphi(\max\{S_b(hx,hx,Ty), S_b(fx,fx,hx), S_b(gy,gy,Ty), \\
& \frac{1}{2}(S_b(hx,hx,gy)+S_b(fx,fx,Ty))\})) \\
=\ & F(\psi(\max\{S_b(x,x,y), S_b(x,x,x), S_b(y,y,y), \frac{1}{2}(S_b(x,x,y)+S_b(x,x,y))\}), \\
& \varphi(\max\{S_b(x,x,y), S_b(x,x,x), S_b(y,y,y), \frac{1}{2}(S_b(x,x,y)+S_b(x,x,y))\})) \\
=\ & F(\psi(S_b(x,x,y)), \varphi(S_b(x,x,y))).
\end{aligned}$$

So, either

$$\psi(S_b(x,x,y)) = 0 \text{ or } \varphi(S_b(x,x,y)) = 0,$$

which further implies that $S_b(x,x,y) = 0$, hence $x = y$. Thus, y is a unique common fixed point of f, g, h and T.

□

If we have $F(s,t) = qs, 0 < q < 1$ and $\psi(t) = t$ in theorem 1 then we have the following corollary as:

Corollary 1. *Suppose that f, g, h and T are self mappings on a S_b-complete metric space (X,S_b) such that $f(X) \subseteq T(X)$,$g(X) \subseteq h(X)$. If*

$$\begin{aligned}
S_b(fx,fx,gy))) \leq\ & \frac{q}{b^4}\max\{S_b(hx,hx,Ty), S_b(fx,fx,hx), S_b(gy,gy,Ty), \\
& \frac{1}{2}(S_b(hx,hx,gy)+S_b(fx,fx,Ty))\}
\end{aligned}$$

holds for all $x,y \in X$ with $0 < q < 1$ and $\frac{2b}{2b^2-1} < 1$ ($\approxeq b > \frac{1+\sqrt{3}}{2}$), then f, g, h and T have a unique common fixed point in X provided that h and T are continuous and pairs $\{f,h\}$ and $\{g,T\}$ are S_{b-}compatible..

Remark 2. *Note that $\frac{2b}{2b^2-1} < 1$ ($\approxeq b > \frac{1+\sqrt{3}}{2}$) holds for $b \geq \frac{5}{4}, \frac{3}{2}, \frac{1+\sqrt{7}}{4}$and in case $b \geq \frac{3}{2}$, we have theorem 2.1 of [18].*

Example 6. *Let $X = [0,1]$ be endowed with S_b-metric*

$$S_b(x,y,z) = (|y+z-2x| + |y-z|)^{\frac{3}{2}},$$

where $b=2$. *Define* f, g, h *and* T *on* X *by* $f(x)=\frac{x^4}{81}$, $g(x)=\frac{x^2}{81}$, $h(x)=\frac{x^2}{9}$, $T(x)=\frac{x}{9}$. *Obviously,* $f(X)\subseteq T(X)$ *and* $g(X)\subseteq h(X)$. *Furthermore, the pairs* $\{f,h\}$ *and* $\{g,T\}$ *are* S_b-*compatible.*

For each x, $y\in X$, *we have*

$$\begin{aligned}
2^4 S_b(fx,fx,gy) &= 2^4(|gy-fx|+|fx-gy|)^{\frac{3}{2}}\\
&= 2^4(2|fx-gy|)^{\frac{3}{2}}\\
&= 2^4 2^{\frac{3}{2}}((\frac{x}{3})^4-(\frac{y}{9})^2)^{\frac{3}{2}}\\
&= 2^4 2^{\frac{3}{2}}((\frac{x}{3})^2+\frac{y}{9})^{\frac{3}{2}}.((\frac{x}{3})^2-\frac{y}{9})^{\frac{3}{2}}\\
&\leq 2^4(\frac{1}{3^2}+\frac{1}{9^2})^{\frac{3}{2}} S_b(hx,hx,Ty)\\
&= \frac{160\times\sqrt{10}}{729} S_b(hx,hx,Ty)\\
&= \frac{S_b(hx,hx,Ty)}{1+(\frac{729}{160\times\sqrt{10}}-1)},
\end{aligned}$$

where $F(s,t)=\frac{s}{1+t}$, $\psi(t)=t$, $\varphi(t)=\frac{729}{160\times\sqrt{10}}-1$, *and* $b=2$. *Thus,* f, g, h *and* T *satisfy all condition of theorem 1. Moreover* 0 *is the unique common fixed point of* f, g, h *and* T.

Corollary 2. *Let* (X,S_b) *be a* S_b*-complete metric space and* $f,g:X\to X$ *are two mappings such that*

$$S_b(fx,fx,gy)\leq\frac{q}{b^4}\max\{S_b(x,x,y),S_b(fx,fx,x),S_b(gy,gy,y),\frac{1}{2}(S_b(x,x,gy)+S_b(fx,fx,y))\},$$

holds for all $x,y\in X$ *with* $0<q<1$ *and* $b\geq\frac{1+\sqrt{3}}{2}$. *Then, there exists a unique point* $y\in X$ *such that* $fy=gy=y$.

Proof. If we take $h=T=I_X$(identity mapping on X) in theorem (2.1), rest of proof is on same lines. □

Corollary 3. *Let* (X,S_b) *be a* S_b*-complete metric space and* $h,T:X\to X$ *are two mappings such that*

$$S_b(x,x,y)\leq q\max\{S_b(hx,hx,Ty),S_b(x,x,hx),S_b(y,y,Ty),\frac{1}{2}(S_b(hx,hx,y)+S_b(x,x,Ty))\},$$

holds for all $x,y\in X$ *with* $0<q<1$. *Then, there exists a unique common fixed point of* h *and* T.

Proof. If we take f and g as identity maps on X, then theorem 1 gives that h and T have a unique common fixed point. □

Corollary 4. *Let (X,S_b) be a S_b-complete metric space and mapping $f: X \to X$ satisfies*

$$S_b(fx,fx,fy) \leq \frac{q}{b^4}\max\{S_b(x,x,y),S_b(fx,fx,x),S_b(fy,fy,y),\frac{1}{2}(S_b(x,x,fy)+S_b(fx,fx,y))\},$$

holds for all $x,y \in X$ with $0<q<1$ and $b \geq \frac{1+\sqrt{3}}{2}$. Then f has a unique fixed point in X.

Proof. Take h and T as identity maps on X and $f=g$ then apply theorem 1 □

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