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New Bounds for the Resolvent Energy of Graphs

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Abstract: The resolvent energy of a graph G of order n is defined as $ER(G) = \sum_{i=1}^{n}(n-\lambda_i)^{-1}$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of G. Lower and upper bounds for the resolvent energy of a graph, which depend on some of the parameters $n, \lambda_1, \lambda_n, \det(\mathscr{R}_A(n)) = \prod_{i=1}^{n} \frac{1}{n-\lambda_i}$, are obtained.

Keywords: resolvent energy, graph, inequalities

1 Introduction

Let M be a square matrix of order n. The resolvent matrix, $\mathscr{R}_M(z)$, of matrix M is defined as [9]

$$\mathscr{R}_M(z) = (zI_n - M)^{-1},$$

where I_n is the unit matrix of order n and z a complex variable. As easily seen, $\mathscr{R}_M(z)$ is also a matrix of order n, that exists for all values of z except when z coincides with an eigenvalue of M.

Let G be the simple graph, A its adjacency matrix and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ eigenvalues of A. The resolvent matrix, $\mathscr{R}_A(z)$, is defined as

$$\mathscr{R}_A(z) = (zI_n - A)^{-1},$$

and its eigenvalues are

$$\frac{1}{z-\lambda_i}, i = 1,2,...,n.$$

Bearing in mind that $\lambda_i \leq n-1$ for all $i = 1,2,...,n$, [4], we could choose $z = n$. Now we have that $\frac{1}{n-\lambda_i}, i = 1,2,...,n$ are the eigenvalues of matrix $\mathscr{R}_A(n) = (nI_n - A)^{-1}$ and $\det(\mathscr{R}_A(n)) = \prod_{i=1}^{n} \frac{1}{n-\lambda_i}$.

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Let G be a graph on n vertices, $n > 1$, with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. Its resolvent energy is [7]

$$ER(G) = \sum_{i=1}^{n} \frac{1}{n - \lambda_i}.$$

Some remarkable properties of $ER(G)$ were revealed in [7]. There are results about defining $ER(G)$ via spectral moments and characteristic polynomial of graphs, and some bounds for the $ER(G)$ in terms of parameters n, m, n_0, where m is the number of edges and n_0 is a nullity of the graph. Additional properties of $ER(G)$ can be also found in the recent papers [1, 5, 6, 11].

In this paper, we obtained some new lower and upper bounds for the resolvent energy of a graph in terms of n, λ_1, λ_n and $\det(\mathscr{R}_A(n))$.

2 Some common inequalities and preliminary lemmas

Now, we introduce some common inequalities which we need for our proofs in the section of main results.

Lemma 2.1 [8] *Let $a_i, r, R \in \mathbb{R}$, $0 < r \le a_i \le R$, $i = 1, ..., n$. Then*

$$n \sum_{i=1}^{n} a_i^2 - \left(\sum_{i=1}^{n} a_i \right)^2 \ge \frac{n}{2}(R - r)^2. \tag{1}$$

Lemma 2.2 [8] *Let $a_i, p_i, r, R \in \mathbb{R}$, $0 < r \le a_i \le R$, $i = 1, ..., n$, $\sum_{i=1}^{n} p_i = 1$. Then*

$$\sum_{i=1}^{n} p_i a_i + rR \sum_{i=1}^{n} \frac{p_i}{a_i} \le r + R. \tag{2}$$

Lemma 2.3 [10] *Let $a_i \in \mathbb{R}^+, i = 1, ..., n$. Then*

$$(n-1) \sum_{i=1}^{n} a_i + n \left(\prod_{i=1}^{n} a_i \right)^{\frac{1}{n}} \ge \left(\sum_{i=1}^{n} \sqrt{a_i} \right)^2 \ge \sum_{i=1}^{n} a_i + n(n-1) \left(\prod_{i=1}^{n} a_i \right)^{\frac{1}{n}}. \tag{3}$$

Lemma 2.4 [8] *Let $a_i, p_i, r, R \in \mathbb{R}$, $0 < r \le a_i \le R$, $i = 1, ..., n$, $\sum_{i=1}^{n} p_i = 1$. Then*

$$\sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} \frac{p_i}{a_i} \le \frac{1}{4} \left(\sqrt{\frac{R}{r}} + \sqrt{\frac{r}{R}} \right)^2. \tag{4}$$

Lemma 2.5 [3] *Let $0 < a_1 \le \cdots \le a_i \le \cdots \le a_k \le \cdots \le a_n$, $p_1, p_2, ..., p_n$ be positive real numbers such that $p_1 + p_2 + \cdots + p_n = 1$ and $Q_i = p_1 + p_2 + \cdots + p_i, R_k = p_k + p_{k+1} + \cdots p_n$. Then*

$$\frac{p_1}{a_1} + \frac{p_2}{a_2} + \cdots + \frac{p_n}{a_n} - \frac{1}{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n} \ge \frac{Q_i R_k (a_k - a_i)^2}{a_i a_k (Q_i a_i + R_k a_k)}, \tag{5}$$

with equality for $a_1 = a_2 = \cdots = a_i, a_k = a_{k+1} = \cdots = a_n, a_{i+1} = a_{i+2} = \cdots = a_{k-1} = \frac{Q_i a_i + R_k a_k}{Q_i + R_k}$.

Lemma 2.6 [2] *Let* $p_1, p_2, ..., p_n$ *be non-negative real numbers and* $a_1, a_2, ..., a_n$ *and* $b_1, b_2, ..., b_n$ *real numbers with the properties* $0 < r_1 \leq a_i \leq R_1 < +\infty$ *and* $0 < r_2 \leq b_i \leq R_2 < +\infty$ *for each* $i = 1, 2, ..., n$. *Further, let* S *be a subset of* $I_n = \{1, 2, ..., n\}$ *which minimizes the expression* $\left| \sum_{i \in S} p_i - \frac{1}{2} \sum_{i=1}^{n} p_i \right|$. *Then*

$$\left| \sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i \right| \leq (R_1 - r_1)(R_2 - r_2) \sum_{i \in S} p_i \left(\sum_{i=1}^{n} p_i - \sum_{i \in S} p_i \right). \quad (6)$$

Lemma 2.7 [4] *A graph has one eigenvalue if and only if it is totally disconnected. A graph has two distinct eigenvalues* $\lambda_1 > \lambda_2$ *with multiplicities* m_1 *and* m_2 *if and only if it consists of* m_1 *complete graphs of order* $\lambda_1 + 1$. *In that case,* $\lambda_2 = -1$ *and* $m_2 = m_1 \lambda_1$.

Lemma 2.8 [8] *Let* $a = (a_i), b = (b_i), c = (c_i)$ *be three sequences of real numbers of the same monotonicity and* $p = (p_i)$ *sequence of real number. Then*

$$\left(\sum_{i=1}^{n} p_i \right)^2 \sum_{i=1}^{n} p_i a_i b_i c_i \geq \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i \sum_{i=1}^{n} p_i c_i. \quad (7)$$

If $a = (a_i)$ *and* $b = (b_i)$ *are oppositely ordered, then the sense of ineqality (7) reverses.*

3 Main results

We represent some new lower and upper bounds for the resolvent energy of graphs.

Theorem 3.1 *Let* G *be a graph on* n *vertices with eigenvalues* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. *Then*

$$ER(G) \geq n(\det(\mathscr{R}_A(n)))^{\frac{1}{n}} + \frac{n}{2(n-1)} \cdot \frac{(\sqrt{n-\lambda_n} - \sqrt{n-\lambda_1})^2}{(n-\lambda_1)(n-\lambda_n)}. \quad (8)$$

Equality is attained if and only if $G = \overline{K}_n$.

Proof. Let's consider inequalities (1) and (3), where we could choose $r = \frac{1}{\sqrt{n-\lambda_n}}, R = \frac{1}{\sqrt{n-\lambda_1}}, a_i = \frac{1}{\sqrt{n-\lambda_i}}, i = 1, ..., n$, to obtain

$$n \sum_{i=1}^{n} \frac{1}{n-\lambda_i} - \left(\sum_{i=1}^{n} \frac{1}{\sqrt{n-\lambda_i}} \right)^2 \geq \frac{n}{2} \left(\frac{1}{\sqrt{n-\lambda_1}} - \frac{1}{\sqrt{n-\lambda_n}} \right)^2. \quad (9)$$

$$\left(\sum_{i=1}^{n} \frac{1}{\sqrt{n-\lambda_i}} \right)^2 \geq \sum_{i=1}^{n} \frac{1}{n-\lambda_i} + n(n-1) \left(\prod_{i=1}^{n} \frac{1}{\sqrt{n-\lambda_i}} \right)^{\frac{1}{n}}. \quad (10)$$

From the definition of the resolvent energy of graph, $ER(G) = \sum\limits_{i=1}^{n} \frac{1}{n-\lambda_i}$, and by (9), we have

$$\begin{aligned} nER &\geq \left(\sum_{i=1}^{n} \frac{1}{\sqrt{n-\lambda_i}}\right)^2 + \frac{n}{2}\left(\frac{1}{\sqrt{n-\lambda_1}} - \frac{1}{\sqrt{n-\lambda_n}}\right)^2 \\ &\geq ER + n(n-1)\left(\prod_{i=1}^{n} \frac{1}{n-\lambda_i}\right)^{\frac{1}{n}} + \frac{n}{2} \cdot \frac{(\sqrt{n-\lambda_n} - \sqrt{n-\lambda_1})^2}{(n-\lambda_1)(n-\lambda_n)} \\ &= ER + n(n-1)(\det(\mathscr{R}_A(n)))^{\frac{1}{n}} + \frac{n}{2} \cdot \frac{(\sqrt{n-\lambda_n} - \sqrt{n-\lambda_1})^2}{(n-\lambda_1)(n-\lambda_n)}, \end{aligned}$$

where in the second inequality we used (10). Now, it follows that

$$ER(G) \geq n(\det(\mathscr{R}_A(n)))^{\frac{1}{n}} + \frac{n}{2(n-1)} \cdot \frac{(\sqrt{n-\lambda_n} - \sqrt{n-\lambda_1})^2}{(n-\lambda_1)(n-\lambda_n)}.$$

If $G = \overline{K}_n$ then $ER(\overline{K}_n) = 1$ and in (8) equality holds.

If equality holds in (8) then equality is attained in (9) i (10), from wich follows that $\lambda_1 = \lambda_2 = \cdots = \lambda_n$. By the Lemma 2.7 it follows that $G = \overline{K}_n$. □

Theorem 3.2 *Let G be a graph on n vertices with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then*

$$ER(G) \geq \frac{n^2 + (n-\lambda_1)(n-\lambda_n)}{n(2n-\lambda_1-\lambda_n)}. \tag{11}$$

Proof. Using Chebyshev inequality for 3 sequences (7) and using inequality (2), for $a_i = \frac{1}{n-\lambda_i}, p_i = \frac{1}{n}, i = 1, ..., n,\ r = \frac{1}{n-\lambda_n}, R = \frac{1}{n-\lambda_1}$ we obtain a lower bound (11). □

Theorem 3.3 *Let G be a graph on n vertices with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then*

$$ER(G) \geq 1 + \frac{(\lambda_n - \lambda_1)^2}{(n-\lambda_1)(n-\lambda_n)(2n-\lambda_1-\lambda_n)} \tag{12}$$

Proof. For $a_i = n - \lambda_i, p_i = \frac{1}{n}, i = 1, ..., n,\ Q_i = R_k = \frac{1}{n}$, the inequality (5) transforms into $ER(G) \geq 1 + \frac{(\lambda_n-\lambda_1)^2}{(n-\lambda_1)(n-\lambda_n)(2n-\lambda_1-\lambda_n)}$. □

Theorem 3.4 *Let G be a graph on n vertices with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then*

$$ER(G) \leq \frac{1}{4} \cdot \frac{(2n-\lambda_1-\lambda_n)^2}{(n-\lambda_1)(n-\lambda_n)}. \tag{13}$$

Proof. Using $p_i = \frac{1}{n}, a_i = \frac{1}{n-\lambda_i}, i = 1, ..., n, r = \frac{1}{n-\lambda_n}, R = \frac{1}{n-\lambda_1}$, in the (4) we obtain the upper bound (13). □

Theorem 3.5 *Let G be a graph on n vertices with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then*

$$-\frac{(\lambda_1-\lambda_n)^2}{(n-\lambda_1)^2(n-\lambda_n)^2}\cdot\frac{n^2}{n-1}\alpha(n)+n\left(\det(\mathscr{R}_A(n))\right)^{\frac{1}{n}} \leq ER(G)$$

$$\leq \frac{(\lambda_1-\lambda_n)^2}{(n-\lambda_1)^2(n-\lambda_n)^2}\cdot n^2\alpha(n)+n\left(\det(\mathscr{R}_A(n))\right)^{\frac{1}{n}}.$$

Proof. The proof follows from the inequality (6) for $a_i = b_i = \frac{1}{\sqrt{n-\lambda_i}}, i = 1,...,n, r_1 = r_2 = \frac{1}{\sqrt{n-\lambda_n}}, R_1 = R_2 = \frac{1}{\sqrt{n-\lambda_1}}$. □

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