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Cacti with a Given Spectral Property

M. Petrović, Z. Radosavljević, M. Rašajski

Abstract: A connected graph G is a cactus if any two of its cycles have at most one common vertex. This article is a survey of results concerning cacti with a given spectral property, categorized as follows: (1) extremal cacti, (2) reflexive cacti.

Keywords: Graph, cactus, index, least eigenvalue, spread, reflexive graphs.

1 Introduction

Let $G = (V(G), E(G))$ be a simple graph with n vertices and $A(G)$ be the $(0,1)$-adjacency matrix of G. Since $A(G)$ is symmetric, its eigenvalues are real. Without loss of generality we can write them in non-increasing order as $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ and call them the eigenvalues of G. The characteristic polynomial $\Phi(G,\lambda)$ of G is defined as $\Phi(G,\lambda) = \det(\lambda I - A(G))$. The family of eigenvalues of a graph G makes up its spectrum. Denote by $\rho(G)$ the largest eigenvalue of G and by $\lambda(G)$ the least eigenvalue of G.

If H is a spanning subgraph of G we shall write $H \subseteq G$; in particular if it is a proper spanning subgraph, we then write $H \subset G$.

Lemma 1 ([6], p. 19) *Let G be a connected graph. If H is connected and $H \subset G$, then $\rho(H) < \rho(G)$.*

Lemma 2 ([6], p. 19) *Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of a graph G and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ the eigenvalues of an induced subgraph H. Then the inequalities*

$$\lambda_{n-m+i} \leq \mu_i \leq \lambda_i \quad (i = 1, \ldots, m)$$

hold.

For $v \in V(G)$, let $G - v$ be the graph that arises from G by deleting the vertex v. The following result is often used to calculate the characteristic polynomials of graphs.

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M. Petrović is with the Faculty of Science, University of Kragujevac, Serbia; Z. Radosavljević, M. Rašajski are with the Faculty of Electrical Engineering, University of Belgrade, Serbia

Lemma 3 ([24]) *Let v be a vertex of G and $\mathscr{C}(v)$ be the set of all cycles of G that contain v. Then*

$$\Phi(G,\lambda) = \lambda\Phi(G-v,\lambda) - \sum_{uv\in E(G)} \Phi(G-u-v,\lambda) - 2\sum_{Z\in\mathscr{C}(v)} \Phi(G-V(Z),\lambda),$$

where $G-V(Z)$ is the graph obtained by removing all the vertices belonging to Z, from G.

In this paper we study cacti with a given spectral property. A connected graph G is a *cactus* if any two of its cycles have at most one common vertex. If all cycles of the cactus G have exactly one common vertex, say v_0, then it is called a *bundle*. The vertex v_0 is called the *central vertex* of the bundle.

2 Extremal cacti

The investigation of graphs with maximal or minimal index among the connected graphs of some classes of graphs is an important topic in the theory of graph spectra. The same holds for the case of maximal or minimal least eigenvalue. Such graphs are called *extremal graphs*.

The problem of determining extremal graphs was proposed by Collatz and Sinogovitz [4] following their investigation of indices of trees. Lovasz and Pelikan [10] proved that among trees with n vertices the star $K_{1,n-1}$ has the largest index ($\sqrt{n-1}$) and the path has the smallest index ($2\cos\frac{\pi}{n+1}$).

In this section we consider the problem of determining extremal graphs in the set of cacti.

Let $C(n)$ be the set of all cacti with n vertices that contain cycles. Denote by $C(n,k)$ the set of all cacti with n vertices and k cycles and by $B(n,k)$ the set of all bundles with n vertices and k cycles. Let $G(C_1,\dots,C_k;r)$ ($k\geq 1$, $r\geq 0$) be the graph depicted in Fig. 1, and let $S(n,k)$ be the set of bundles in $B(n,k)$ of the form $G(C_1,\dots,C_k;r)$.

Let $B^*(n,k)$ be the set of bundles in $S(n,k)$ for which the length of the longest cycle is less than or equal to 4. Also, let t be the maximal number of cycles of length 4; it is easy to see that $t=\min\{k,n-(2k+1)\}$. Denote by $B_{k,s}=B(k-s,s,n-(2k+s+1))$ ($s=0,1,\dots,t$) the graph in $B^*(n,k)$ which contains $k-s$ cycles of length 3, s cycles of length 4 and $n-(2k+s+1)$ attached vertices of degree one at v_0 (Fig. 2). Then $B^*(n,k)=\{B_{k,0},\dots,B_{k,t}\}$.

Table 1 contains the list of all bundles in $B^*(n,k)$ ($3\leq n\leq 7$, $1\leq k\leq \frac{1}{2}(n-1)$). For each bundle the least eigenvalue is given in this table (see [5], [6] and [7]).

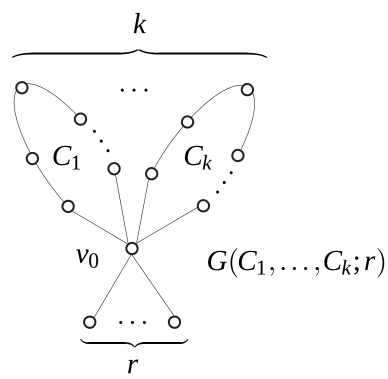

Fig. 1. The bundle $G(C_1,\dots,C_k;r)$

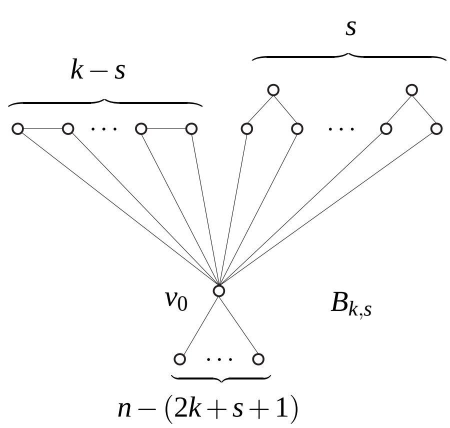

Fig. 2. The bundle $B_{k,s}$

2.1 Cacti with maximal index

First, we consider the maximal index problem in the set of cacti. Let B^* have the maximal index in $C(n,k)$ $(k \geq 1, n \geq 2k+1)$. The procedure of determining the graph B^* is described in [1] and it consists of three steps.

In the first step it is proved that any two cycles of the graph B^* have a common vertex and any three cycles of the graph B^* have exactly one common vertex. Now, we conclude that all cycles of the graph B^* have exactly one common vertex v_0, i.e. they form a bundle.

In the second step it is proved that B^* is a bundle with unique tree attached to the vertex v_0, and this tree contains only vertices at distance one from v_0. It follows that $B^* \in S(n,k)$.

Finally, it is proved that all cycles of B^* have length three, i.e. $B^* = B(k,0,n-(2k+1))$.

Let $m = \lfloor \frac{n-1}{2} \rfloor$. So, the following theorem holds.

Theorem 1 *([1]) The graph $B(k,0,n-(2k+1))$ is the unique graph with maximal index in the set of all cacti with n vertices and k cycles ($1 \leq k \leq m$).*

n	k	The graph	The least eigenvalue
3	1	$B(1,0,0)$	-1,00000
4	1	$B(1,0,1)$	-1,48119
		$B(0,1,0)$	-2,00000
5	1	$B(1,0,2)$	-1,81361
		$B(0,1,1)$	-2,13578
	2	$B(2,0,0)$	-1,56155
6	1	$B(1,0,3)$	-2,08613
		$B(0,1,2)$	-2,28825
	2	$B(2,0,1)$	-1,90321
		$B(1,1,0)$	-2,19117
7	1	$B(1,0,4)$	-2,32340
		$B(0,1,3)$	-2,44949
	2	$B(2,0,2)$	-2,17741
		$B(1,1,1)$	-2,35269
		$B(0,2,0)$	-2,44949
	3	$B(3,0,0)$	-2,00000

Table 1. The bundles $B^*(n,k)$ $(3 \leq n \leq 7,\ 1 \leq k \leq \frac{1}{2}(n-1))$

Theorem 2 *([1]) The graph $B(m,0,n-(2m+1))$ is the unique graph with maximal index in the set $C(n)$ $(n \geq 3)$.*

Proof. Let B^* have the maximal index in $C(n)$ $(n \geq 3)$. By Theorem 1 we may assume that

$$B^* \in \bigcup_{k=1}^{m} B(k,0,n-(2k+1)).$$

It is obvious that

$$B(1,0,n-3) \subset B(2,0,n-5) \subset \cdots \subset B(m,0,n-(2m+1)).$$

By Lemma 1 we have

$$\rho(B(1,0,n-3)) < \rho(B(2,0,n-5)) < \cdots < \rho(B(m,0,n-(2m+1))).$$

So, we obtain that the graph $B(m,0,n-(2m+1))$ depicted in Fig. 2, is the graph with maximal index in the set $C(n)$ of all cacti with n vertices. This completes the proof. □

In the paper [8] the maximal index problem in the set of cacti with perfect matchings is considered. Recall, a *perfect matching* of G is a set of mutually independent edges that cover every vertex of G.

Denote by $C_1(2n,k)$ the set of all cacti on $2n$ vertices and k cycles, having a perfect matching. Also, denote by $G_{n,k}$ the bundle in $B(2n,k)$ that contains k cycles of length 3 and a pendant edge together with $n-k-1$ hanging paths of length two attached to v_0 (Fig. 3).

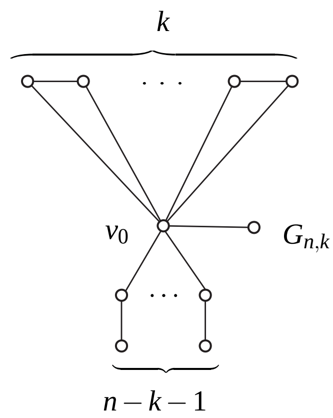

Fig. 3. The bundle $G_{n,k}$

Theorem 3 ([8],[3]) *Let $G \in C_1(2n,k)$. If $n \geq 4$ and $k \geq 1$ then*

$$\rho(G) \leq \rho(G_{n,k}),$$

with equality if and only if $G = G_{n,k}$.

For $k = 0$ the corresponding class is the set of all trees. This case is covered by the following result from literature.

Theorem 4 ([2],[27]) *Let T be a tree in $C_1(2n,0)$. Then $\rho(T) \leq \rho(G_{n,0})$ and the equality holds if and only if $T = G_{n,0}$.*

Finally, observe that

$$G_{n,0} \subset G_{n,1} \subset \cdots \subset G_{n,n-1}$$

and by Lemma 1 we have

$$\rho(G_{n,0}) < \rho(G_{n,1}) < \cdots < \rho(G_{n,n-1}).$$

So, we also have

Theorem 5 *The graph $G_{n,n-1}$ has maximal index in the set of all cacti on $2n$ vertices having perfect matchings.*

2.2 Cacti with minimal least eigenvalue

Now, we consider the minimal least eigenvalue problem in the set of cacti. Let B^{**} have the minimal least eigenvalue in $C(n,k)$ $(k \geq 1,\ n \geq 2k+1)$.

Let $X = (x_1, x_2, \ldots, x_n)^T$ be a column vector in $\mathbb{R}^n$, and G a graph on vertices $v_1, v_2, \ldots, v_n$. Then X can be considered as a function defined on the vertex set of G, that is, for any vertex v_i, we map it to $x_i = X(v_i)$, sometimes written x_{v_i}. We say that x_i is the value on the vertex v_i given by X. The following theorem has an important role in determining the graph B^{**}.

Theorem 6 *([15]) Let B be a bundle of the form $G(C_1,\dots,C_k;r)$ ($k\ge 1$, $r\ge 0$) (Fig. 1). If the least eigenvalue $\lambda(B)<-2$, then for any eigenvector X of $\lambda(B)$ the relation $x_{v_0}=x_0\neq 0$ is satisfied. Also, for any cycle $C=v_0v_1\dots v_{l-1}v_0$ of B the following properties hold, with $s=\lfloor\frac{l}{2}\rfloor$:*

$$x_i = x_{l-i} \quad (i=1,2,\dots,l-1) \tag{1}$$

$$|x_0|>|x_1|>\dots>|x_{s-1}|>|x_s|>0 \tag{2}$$

$$x_{i-1}x_i<0 \quad (i=1,\dots,s). \tag{3}$$

The procedure of determining the graph B^{**} is described in [15] and it consists of three steps.

In the first step it is proved that for any graph $G\in C(n,k)\backslash S(n,k)$ there is a graph $B\in S(n,k)$ such that $\lambda(B)<\lambda(G)$ and G,B have the same cycle lengths. Now, without loss of generality, we may assume that $B^{**}\in S(n,k)$ $(k\ge 1, r\ge 0)$.

Denote by $c(G)$ the *circumference* of G, i.e. the length of the longest cycle in G. In the second step it is proved that $c(B^{**})\le 4$, i.e. $B^{**}\in B^*(n,k)$. Precisely, the following theorem holds.

Theorem 7 *([15]) If B^{**} has the minimal least eigenvalue in $C(n,k)$ $(k\ge 1, n\ge 2k+1)$, then $B^{**}\in B^*(n,k)$.*

At the end, in the third step, the following theorem is proved.

Theorem 8 *([15]) Let B^{**} have the minimal least eigenvalue in $C(n,k)$ $(k\ge 1$, $n\ge 2k+1)$ and $t=\min\{k,n-(2k+1)\}$. Also, let $n_0(k)=12$ for $1\le k\le 2$, $n_0(k)=13$ for $3\le k\le 4$ and $n_0(k)=14$ for $5\le k\le 6$.*

1^0 If $k\le 6$ and $n<n_0(k)$, then $B^{**}=B(k-t,t,n-(2k+t+1))$.

2^0 If $k\le 6$ and $n\ge n_0(k)$ or $k\ge 7$, then $B^{**}=B(k,0,n-(2k+1))$.

Theorem 9 *([15]) Let B^{**} have the minimal least eigenvalue in $C(n)$ $(n\ge 3)$.*

1^0 If $n=3$ or $n\ge 12$, then $B^{**}=B(1,0,n-3)$.

2^0 If $4\le n\le 6$ or $8\le n\le 11$, then $B^{**}=B(0,1,n-4)$.

3^0 If $n=7$, then $B^{**}=B(0,1,3)$ or $B^{**}=B(0,2,0)$.

Proof. Let B^{**} have the minimal least eigenvalue in $C(n)$ $(n\ge 3)$. Also, let $m=\lfloor\frac{1}{2}(n-1)\rfloor$ and $l=\lfloor\frac{1}{3}(n-1)\rfloor$. Then

$$C(n)=\bigcup_{k=1}^{m} C(n,k)$$

and by Theorem 7 we may assume that

$$B^{**}\in\bigcup_{k=1}^{m} B^*(n,k),$$

where $B^*(n,k)=\{B_{k,0},\dots,B_{k,t}\}$.

If $n \leq 7$, the results follow from the Table 1.

Now, let $n \geq 8$. Then, there is a bundle $B \in B^*(n,k)$ such that $B(0,2,1)$ is an induced subgraph of B. By Lemma 2 we conclude that $\lambda(B^{**}) \leq \lambda(B) \leq \lambda(B(0,2,1)) = -2,58874$.

By applying Lemma 3 to the vertex v_0 of $B_{k,s}$ we obtain

$$\Phi(B_{k,s},\lambda) = \lambda^{n-2k-2}(\lambda-1)^{k-s-1}(\lambda+1)^{k-s}(\lambda^2-2)^{s-1}Q(k,s,\lambda)\,,$$

where

$$\begin{aligned} Q(k,s,\lambda) &= \lambda^5-\lambda^4-(n-s+1)\lambda^3+(n-2k+s+1)\lambda^2 \\ &+ (2n-6s-2)\lambda-2(n-2k-s-1)\,. \end{aligned} \tag{4}$$

From (4) we get

$$Q(k,s,\lambda)-Q(k+1,s,\lambda) = 2(\lambda^2-2) > 0$$

and we conclude that the inequalities

$$\lambda(B_{1,0}) < \lambda(B_{2,0}) < \cdots < \lambda(B_{m,0})$$

hold.

Analogously, from (4) we obtain

$$Q(k,s,\lambda)-Q(k+1,s+1,\lambda) = (\lambda-1)(6-\lambda^2) > 0\,.$$

Now, if $2 \leq k \leq l$ then $t = k$ and we see that the inequalities

$$\lambda(B_{k,k}) > \lambda(B_{k-1,k-1}) > \cdots > \lambda(B_{1,1}) \tag{5}$$

are satisfied. Also, we conclude that for $k > l$ and $t > 0$ the inequalities

$$\lambda(B_{k,t}) > \lambda(B_{k-1,t-1}) > \cdots > \lambda(B_{k-t,0}) \geq \lambda(B_{1,0}) \tag{6}$$

hold.

Finally, from (5) and (6) we have that $B^{**} \in B^*(n,1)$. In conjunction with Theorem 8 this result completes the proof. □

2.3 Cacti with maximal spread

Recall that the *spectral spread* $sp(G)$ of a graph G of order n is equal to $\lambda_1(G)-\lambda_n(G)$; so $sp(G) = \rho(G)-\lambda(G)$. Based on Theorems 1 and 8 we immediately get the following result.

Corollary 1 *Let $n_0(k) = 12$ for $1 \leq k \leq 2$, $n_0(k) = 13$ for $3 \leq k \leq 4$ and $n_0(k) = 14$ for $5 \leq k \leq 6$. If $k \leq 6$ and $n \geq n_0(k)$ or $k \geq 7$, then for each $G \in C(n,k)$*

$$sp(G) \leq sp(B(k,0,n-(2k+1))$$

with equality if and only if $G = B(k,0,n-(2k+1))$ (see Fig. 2).

At the end of this section we quote two open problems.

Problem 1. Determine graph(s) with minimal index among all graphs in $C(n,k)$.

Problem 2. Determine graph(s) with maximal least eigenvalue among all graphs in $C(n,k)$.

3 Reflexive cacti

Graphs having $\lambda_2 \leq 2$ are usually called reflexive graphs. In fact, these graphs correspond to sets of vectors in the Lorentz space $R^{p,1}$ having Gram matrix $2I - A$, and consequently norm 2 and mutual angles 90° and 120°. They are Lorentzian counterparts of some graphs that occur in the theory of reflexion groups and they have direct application to the construction and classification of such groups [14]. Graphs having $\lambda_2 \leq 2 \leq \lambda_1$ are also known as hyperbolic graphs.

Since the property $\lambda_2 \leq 2$ is clearly a hereditary one, any search for reflexive graphs (within a prescribed class of graphs) is naturally aimed at finding maximal connected reflexive graphs. The crucial role in those investigations and in describing of such maximal graphs play well known Smith graphs ([25], [6] p. 79), the only connected graphs having the index $\lambda_1 = 2$ (Fig. 4).

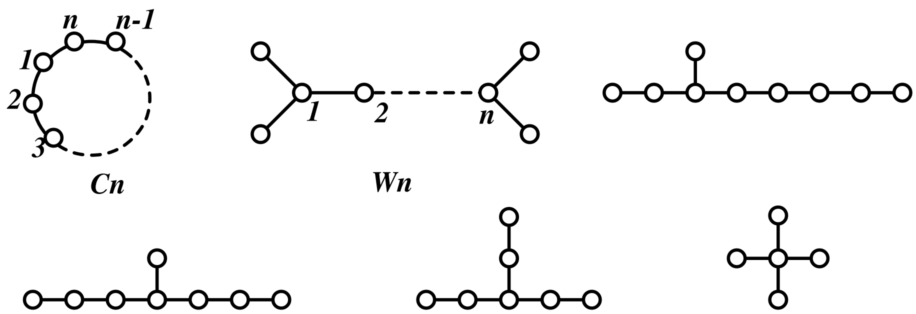

Fig. 4. Smith graphs

The first two Smith graphs (cycles C_n and double-head snakes W_n) can have arbitrary number of vertices; the last one (the star $K_{1,4}$) is actually W_1, but it is often convenient to consider it as a separate case.

In what follows, when saying "subgraph" we mean induced subgraph of a graph. Then, G is a supergraph of H if H is a subgraph of G.

3.1 Some general results. Trees

Reflexive trees were investigated in [11], while [13] contains some more general results concerning the second largest eigenvalue of a tree that can directly be applied to a given λ_2. Following the terminology and designations of this paper let us consider a graph of the shape displayed in Fig. 5 and let it be denoted by (G_1, x_1, x_2, G_2). (This means that G_1 and G_2 are two disjoint graphs, $x_1 \in G_1$, $x_2 \in G_2$ and the whole graph is obtained by adding a single edge between the vertices x_1 and x_2.)

We will call a graph G λ -critical at vertex x if $\lambda_1(G-x) < \lambda < \lambda_1(G)$. A graph (G_1, x_1, x_2, G_2) will be called a λ-twin if G_i is λ -critical at x_i, $i = 1, 2$. Also, let us call a tree T λ-trivial if there is a vertex $x \in T$ such that $\lambda_1(T-x) \leq \lambda$, which means that the

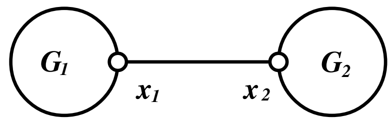

Fig. 5.

largest eigenvalue of no component of $T-x$ exceeds λ. A subtree T_1 of a tree T is called extremal if, for some vertex $x \in T$, T_1 is a component of $T-x$.

Lemma 4 ([13]) *If a tree T with $\lambda_2(T) \leq \lambda$ contains extremal subtrees with eigenvalue λ, then T is λ-trivial.*

Lemma 5 ([13]) *A tree T with $\lambda_2(T) \leq \lambda$ is either λ-trivial or a λ-twin.*

Lemma 6 ([13]) *Let T be a λ-trivial tree and x a vertex of T such that $\lambda_1(T-x) \leq \lambda$. Then $\lambda_2(T) = \lambda$ if and only if k ($k \geq 2$) components of $T-x$ have the largest eigenvalue λ; in this case x is a special vertex of T, and λ is an eigenvalue of T of multiplicity $k+1$.*

With one exception all Smith graphs are trees, and by putting $\lambda = 2$ in Lemma 4 and 5, we immediately get the following consequences.

Corollary 2 *If a reflexive tree contains extremal subtrees which are Smith trees, then T is 2-trivial. Also, a reflexive tree is either 2-trivial or a 2-twin.*

Clearly, a tree T cannot be 2-trivial and a 2-twin at the same time and also T can be represented in at most one way as a 2-twin, and these facts hold for arbitrary λ, too.

The next theorem was crucial in directing further investigations towards finding various classes of maximal reflexive graphs.

Let us call a connected graph positive, null or negative depending on whether its index is greater than, equal to or less than 2, respectively (i.e. whether it is a proper supergraph of a Smith graph, a Smith graph or a proper subgraph of a Smith graph).

Theorem 10 *([22], [16]) Let G be a graph with a cut-vertex v. Then we have: 1) if at least one component of $G-v$ is positive and at least one more is non-negative, then $\lambda_2(G) > 2$; 2) if at least two components of $G-v$ are null and all other are non-positive, $\lambda_2(G) = 2$; 3) if at most one component of $G-v$ is null and any other is negative, $\lambda_2(G) < 2$.*

This theorem evidently points at cacti as a convenient class of graphs for the search for maximal reflexive graphs but, on the other hand, it has no answer what happens if one component is positive while all others are negative. That is why just such cases are the subject of further consideration.

3.2 Bicyclic graphs with a bridge. Pouring of Smith trees

An early result on maximal reflexive graphs is the case of bicyclic graphs whose two cycles are joined by a single edge - the bridge. Let the graph of Fig. 5 be such that G_1 and G_2 are unicyclic graphs and let x_1 and x_2 belong to the cycles. If a vertex v of a cycle is of degree $deg(v) > 2$, we say that v is loaded, and the same holds for x_1 or x_2 if their degrees are greater than 3. If e.g. G_1 has no loaded vertices, by applying Theorem 10 to the vertex x_2 we can easily establish whether the graph is reflexive. That is why we assume that both sides (G_1 and G_2) have at least one loaded vertex. On this assumption, all maximal reflexive graphs of the described shape were found in [22]. Among them, a specially interesting case is that when we assume that the only loaded vertices are x_1 and x_2 (and then we say that the two cycles are free). To describe this result, we have to explain the notion of pouring of a tree (see Fig. 6).

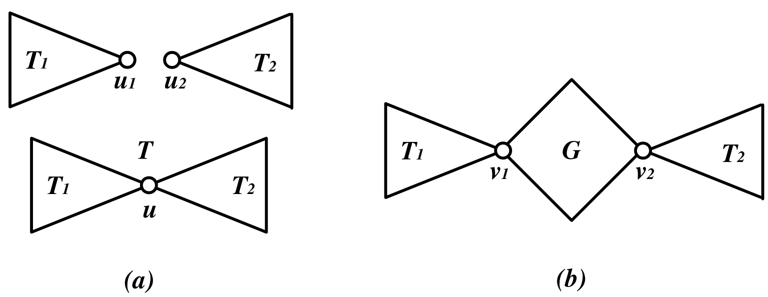

Fig. 6.

If we form a tree T by identifying vertices u_1 and u_2 ($u_1 = u_2 = u$) of two trees T_1 and T_2, respectively, we may say that the tree T can be split at its vertex u into T_1 and T_2. If we split a tree T at all its vertices u, in all possible ways, and in each case attach the parts at splitting vertices u_1 and u_2 to some vertices v_1 and v_2 of a graph G (i.e. identify u_1 with v_1 and u_2 with v_2, and vice versa), we say that in the obtained family of graphs the tree T is pouring between the vertices v_1 and v_2. Of course, this description includes also attaching of the intact tree T, at each vertex, to v_1 and v_2.

Theorem 11 *([21], [22]) If G is a bicyclic graph with a bridge between its cycles to which Theorem 10 cannot be applied and if all vertices of the two cycles except c_1 and c_2 are not loaded, then G is reflexive if and only if it is a subgraph of some of the graphs of Fig. 7(a) (obtained by pouring of Smith trees between the vertices c_1 and c_3), or of the graph of Fig. 7(b).*

Of course, S_1 and S_2 are parts (obtained by splitting) of a Smith tree, which includes the possibility of attaching an intact Smith tree at c_1. But a cycle is also a Smith graph, and it turned out indeed that in such a way we also come to a maximal reflexive graph.

Theorem 12 *([21], [22]) In the class of reflexive cacti that have a bridge between some of its cycles and to which Theorem 10 cannot be applied there is only one family of graphs*

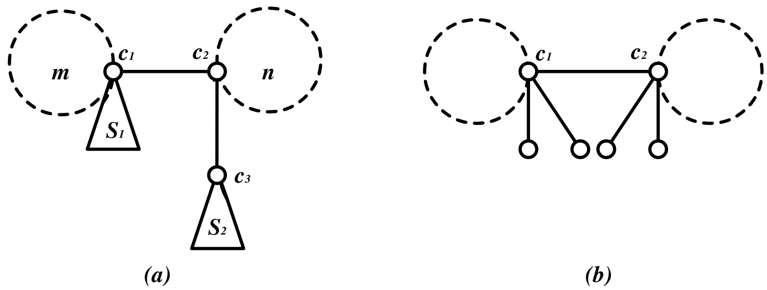

Fig. 7.

with more than two cycles; it is the family of tricyclic graphs T_0 of Fig. 8 and these graphs are maximal within the scope of the described class.

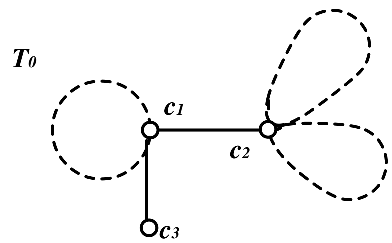

Fig. 8.

3.3 Maximum number of cycles. Replacement

Theorem 10 points out that a reflexive cactus whose cycles have the common vertex, i.e. form a bundle, can have an arbitrary number of cycles. To look for maximal reflexive cacti in this case is an extremely hard problem. Also, Theorem 10 directs further investigations to those cacti in which we have one proper supergraph and all other proper subgraphs of Smith graphs after the decomposition of the starting graph. Therefore, in what follows we are looking for reflexive cacti on the two assumptions: that they are not bundles and that Theorem 10 cannot answer the question whether they are reflexive. The technique applied to all those investigations is a combination of various elements of algebraic apparatus of spectral graph theory (especially [24]). For computations we use the expert system GRAPH and, recently, the newer version newGRAPH [26].

Theorem 13 *([21]) A cactus to which Theorem 10 cannot be applied and whose cycles do not form a bundle has at most five cycles. The only such graphs with five cycles, which are all maximal, are the four families of cacti of Fig. 9.*

These four families, Q_1, Q_2, T_1 and T_2, are now the starting point for finding maximal reflexive cacti with less than five cycles (always on the mentioned two conditions). That is why the next theorem is of great importance.

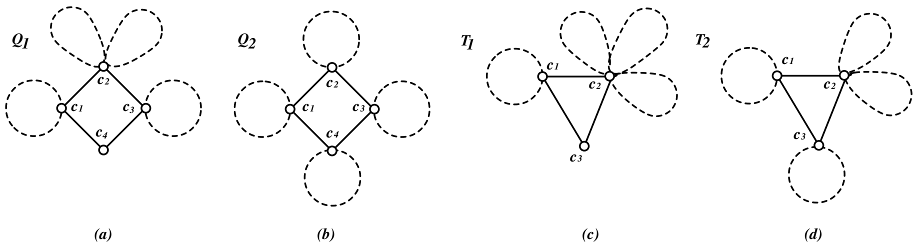

Fig. 9.

Theorem 14 *([19]) Suppose that a graph of the form shown in Fig. 10(a) (an arbitrary cycle C attached to an arbitrary cactus G at vertex v) is a maximal reflexive cactus for which* $\Phi(2) = 0$ *and* $\Phi(G,2) < 0$ *and for any extension* G_1 *formed by attaching to G a pendant edge* $\Phi(G_1,2) - 2\Phi(G_1 - v,2) > 0$ *holds. If the free cycle C is replaced by an arbitrary Smith tree (Fig. 10(b)), then the resulting graph is again a maximal reflexive cactus.*

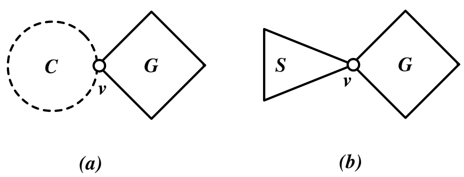

Fig. 10.

3.4 Maximal reflexive cacti with less than five cycles

Of course, Theorem 14 itself gives many possibilities of replacing free cycles of the families of graphs in Fig. 9 by Smith trees, giving rise to maximal reflexive cacti with less cycles. But in fact, there are many more possibilities for their construction. All maximal reflexive cacti with four cycles (always on the mentioned two conditions) have been found in [20] and [21], and we can point out here only some interesting parts of the result. Thus, if we remove the free cycle at c_4 of the graph Q_2 and if Smith trees pour between c_2 and c_4, all such graphs are maximal reflexive cacti. The same happens if we delete cycle at c_3 of T_2 and Smith trees pour between c_2 and c_3. However, if we remove one of the two free cycles at c_2 of T_2, again Smith trees pour between c_2 and c_3 and these graphs are maximal except in one case, which gives rise to three new maximal reflexive cacti ([20], [18]).

Further deletion of cycles from the graphs of Fig. 9 opens possibilities of constructing tricyclic maximal reflexive cacti. It turned out [12] that the starting cyclic structure (which, after various ways of extension, will give rise to maximal graphs) may suitably be separated into four families of graphs, displayed in Fig. 11.

It is evident (when compared with the graphs of Fig. 9) that cases (a), (b) and (c) actually allow attaching of new free cycles, which then, in tricyclic case, can be replaced by Smith trees, while case (d) allows only attaching of trees. In [23], case (d) is completely

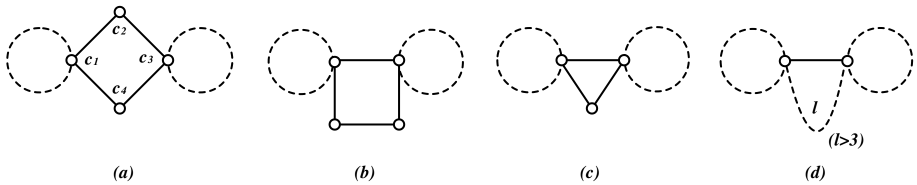

Fig. 11.

solved. For cases (a) and (c), so far we have only parts of the solution. In case (a), we obtain a class of maximal reflexive graphs if all possible pairs of Smith trees pour between c_2 and c_4 ([12]). Similarly, in case (c), pairs of Smith trees pour between c_1 and c_2, giving maximal graphs except in two cases ([18]).

Besides the initial case solved in [22], bicyclic reflexive cacti have been considered in [19]. Some of them can be constructed by means of Theorem 14, but the most remarkable case is the following. Let us start from the graph Q_1, remove all three free cycles at vertex c_2 and let three Smith trees pour between c_2 and c_3. Then, these graphs are maximal reflexive cacti except in three cases, each of which then gives rise to three new maximal cacti ([19]).

Unicyclic reflexive graphs have been treated in [17] and [9]. Besides one more very interesting case of pouring of pairs of Smith trees, in [17] it has been established the maximum number of loaded vertices of the cycle in these graphs.

Theorem 15 *([17]) The cycle of a unicyclic reflexive graph has at most* 8 *loaded vertices. This number is attained only if the cycle is octagon and it gives rise to* 6 *maximal graphs.*

Also, the length of the cycle with 7 loaded vertices is at most 10 ([9]). This paper also contains all maximal graphs with 7 loaded vertices if the length of the cycle is 10, 9 and 8.

At the end, let us notice that the role of Smith trees in the construction and description of maximal reflexive cacti is not at all exhausted by their pouring. The doctoral thesis [23] contains some other interesting cases of application of these graphs.

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References

[1] B. Borovićanin, M. Petrović, *On the index of cactuses with n vertices*, Publ. Inst. Math. (Beograd) 79 (93) (2006), 13-18.

[2] A. Chang, *On the largest eigenvalue of a tree with perfect matching*, Discrete Math., 269 (2003), 45-63.

[3] A. Chang, F. Tian, *On the spectral radius on unicyclic graphs with perfect matching*, Linear Algebra Appl., 370 (2003), 237-250.

[4] L. Collatz, U. Sinogowitz, *Spektren endlicher Grafen*, Abh. Math. Sem. Univ. Hamburg 21(1957), 63-77.

[5] D. Cvetković, M. Doob, I. Gutman, A. Torgašev, *Recent Results in the Theory of Graph Spectra*, North-Holland (Amsterdam), 1988.

[6] D. Cvetković, M. Doob, H. Sachs, *Spectra of graphs - Theory and Applications*, Deutscher Verlag der Wissenschaften, Academic Press, Berlin, New York, 1980 (third ed., Johann Ambrosius Barth, Heidelberg, Leipzig, 1995).

[7] D. Cvetković, M. Petrić, *A table of connected graphs on six vertices*, Discrete Math. 50(1984), 37-49.

[8] Z. Huang, H, Deng, S. Simić, *On the spectral radius of cactuses with perfect matchings*, Appl. Anal. Discrete Math. 5(2011), 14-21.

[9] T. Koledin, Z. Radosavljević, *Unicyclic reflexive graphs with seven loaded vertices of the cycle*, Filomat 23:3 (2009), pp. 253-264.

[10] L. Lovasz, J. Pelikan, *On the eigenvalues of trees*, Periodica Math. Hung. 3(1973), 175-182.

[11] G. Maxwell, *Hypebolic trees*, J. Algebra 54 (1978), 46-49.

[12] B. Mihailović, Z. Radosavljević, *On a class of tricyclic reflexive cactuses*, Univ. Beograd, Publ. Elektrotehn. Fak., Ser. Mat. 16 (2005), 55-63.

[13] A. Neumaier, *The second largest eigenvalue of a tree*, Linear Algebra and its Appl., 46 (1982), 9–25.

[14] A. Neumaier, J. J. Seidel, *Discrete hyperbolic geometry*, Combinatorica, 3 (2) (1983), 219-237.

[15] M. Petrović, T. Aleksić, V. Simić, *On the least eigenvalue of cacti*, Linear Algebra Appl. (2011), doi:10.1016/j.laa.2011.02.011.

[16] M. Petrović, Z. Radosavljević, *Spectrally constrained graphs*, Fac. of Science, Kragujevac, Serbia, 2001.

[17] Z. Radosavljević, *On unicyclic reflexive graphs*, Applicable Analysis and Discrete Mathematics 1 (2007), pp. 228-240.

[18] Z. Radosavljević, B. Mihailović, M. Rašajski, *Decomposition of Smith graphs in maximal reflexive cacti*, Discrete Mathematics, Volume 308, Issues 2-3, 6 Feb 2008, 355-366.

[19] Z. Radosavljević, B. Mihailović, M. Rašajski, *On bicyclic reflexive graphs*, Discrete Mathematics, Volume 308, Issues 5-6, 28 March 2008, Pages 715-725.

[20] Z. Radosavljević, M. Rašajski, *A class of reflexive cacti with four cycles*, Univ. Beograd, Publ.Elektrotehn.Fak., Ser.Mat., 14 (2003), 64–85.

[21] Z. Radosavljević, M. Rašajski, *Multicyclic treelike reflexive graphs*, Discrete Math., Vol. 296/1 (2005), 43-57.

[22] Z. Radosavljević, S. Simić, *Which bicyclic graphs are reflexive?*, Univ. Beograd, Publ. Elektrotehn. Fak., Ser. Mat. 7 (1996), 90-104.

[23] M. Rašajski, *Višekonturni refleksivni grafovi*, PhD thesis, Mat. Fak. Univ. u Beogradu, 2006.

[24] A. Schwenk, *Computing the charateristic polynomial of a graph*, in:R. Bari, F. Harary (Eds.), Graphs and Combinatorics - Lecture Notes in Mathematics, vol. 406, Springer-Verlag, Berlin, Heidelrberg, New York, 1974, 153-172.

[25] J. H. SMITH, *Some properties of the spectrum of a graph*, In: Combinatorial Structures and Their Applications. Ed. R. Guy, H. Hanani, N. Sauer, J. Schonheim. Gordon and Breach, Science Publ., Inc., New York–London–Paris 1970, 403–406.

[26] D. STEVANOVIĆ, V. BRANKOV, D. CVETKOVIĆ, S. SIMIĆ, *newGRAPH, the expert system*, http://www.mi.sanu.ac.rs/newgraph.

[27] G. XU, *On the spectral radius of trees with perfect matchings*, Combinatoris and Graph theory, World Scientific, Singapure, 1997.