SCIENTIFIC PUBLICATIONS OF THE STATE UNIVERSITY OF NOVI PAZAR
SER. A: APPL. MATH. INFORM. AND MECH. vol. 3, 2 (2011), 117-121.

General Solutions of System of Finite Equations

D. Banković

Abstract: In this paper we research the system of finite equations. We give the consistency condition and we determine the general solution of this system.

Keywords: Finite equation, Boolean equation, general solution

We state the definition of general solution and reproductive general solution of an equation.

Definition 1. Let E be a given non-empty set and R be a given unary relation of E. A formula $x = g(t)$, where $g : E \to E$ is a given function, represents a general solution of the equation $R(x)$ if and only if

$$(\forall t)R(g(t)) \wedge (\forall x)(R(x) \Rightarrow (\exists t)x = g(t)).$$

Definition 2. A formula $x = g(t)$, where $g : E \to E$ is a given function, represents a reproductive general solution of the equation $R(x)$ if and only if

$$(\forall t)R(g(t)) \wedge (\forall t)(R(t) \Rightarrow t = g(t)).$$

One can prove that the last formula is equivalent to

$$(\forall t)(R(t) = 0 \Leftrightarrow t = g(t)).$$

Let $Q = \{q_0, q_1, \ldots, q_p\}$ be a given set of $p+1$ elements and $M = \{0,1\}$. Define the operations $+$, $\cdot$ and x^y in the following way:

$+$	0	1
0	0	1
1	1	1

$\cdot$	0	1
0	0	0
1	0	1

$$x^y = \begin{cases} 1, & \text{if} \quad x = y \\ 0, & \text{otherwise} \end{cases} \quad (x, y \in Q \cup \{0,1\}).$$

Manuscript received November 15, 2010; revised February 22, 2010; accepted March 30, 2011.
Dragić Banković is with University of Kragujevac, Faculty of Science, Kragujevac, Serbia

Assuming that

$$(\forall x \in \{0,1\} \cup Q)(x+0 = x \wedge 0+x = x \wedge x \cdot 0 = 0 \wedge x \cdot 1 = x \wedge 1 \cdot x = x)$$

S. Prešić considered, in [2], the following $x-$equation, called finite equation,

$$a_0 \cdot x^{q_0} + a_1 \cdot x^{q_1} + \cdots + a_p \cdot x^{q_p} = 0, \tag{1}$$

where $a_i \in \{0,1\}$, $x \in Q$. It is obvious that an element q_i is a solution of (1) if and only if $a_i = 0$. Equation (1) is consistent (has a solution) if and only if $a_0 \cdot a_1 \cdots a_m = 0$. Prešić described all the reproductive general solutions of (1). In the paper [1] all the general solutions of (1) were described. More results of finite equations can be found in [4].

Theorem 1 *([1]) If equation (1) is consistent, then a formula $x = A(t)$ represents a general solution of (1) if and only if $A(t)$ is of the form*

$$\begin{aligned} A(t) &= \sum_{k=0}^{p} (a^0_{i_{k,0}} q_{i_{k,0}} + a_{i_{k,0}} a^0_{i_{k,1}} q_{i_{k,1}} + a_{i_{k,0}} a_{i_{k,1}} a^0_{i_{k,2}} q_{i_{k,2}} + \cdots + a_{i_{k,0}} a_{i_{k,1}} \cdots a^0_{i_{k,p-2}} q_{i_{k,p-2}} \\ &+ a_{i_{k,0}} a_{i_{k,1}} \cdots a^0_{i_{k,p-1}} q_{i_{k,p-1}} + a_{i_{k,0}} a_{i_{k,1}} \cdots a_{i_{k,p-1}} q_{i_{k,p}}) t^{q_k} \end{aligned} \tag{2}$$

where

$$(i_{k,0}, i_{k,1}, \ldots, i_{k,p}) \quad \text{are permutations of} \quad \{0,1,\ldots,p\}$$

and

$$(i_{0,0}, i_{1,0}, \ldots, i_{p,0}) \quad \text{is a permutation of} \quad \{0,1,\ldots,p\}.$$

If $(i_{0,0}, i_{1,}, \ldots, i_{p,0}) = (0,1,\ldots,p)$ then the solution (2) is reproductive.

Now we consider the system of m Prešić's equations

$$\begin{aligned} & a_{1,0}x^{q_0} + a_{1,1}x^{q_1} + \cdots + a_{1,p}x^{q_p} = 0 \\ & \wedge a_{2,0}x^{q_0} + a_{2,1}x^{q_1} + \cdots + a_{2,p}x^{q_p} = 0 \\ & \vdots \\ & \wedge a_{m,0}x^{q_0} + a_{m,1}x^{q_1} + \cdots + a_{m,p}x^{q_p} = 0. \end{aligned} \tag{3}$$

It is obvious that an element q_i is a solution of system (3) if and only if $a_{1,i} = a_{2,i} = \ldots = a_{m,i} = 0$.

Theorem 2 *Sistem (3) is consistent (has a solution) if and only if*

$$\prod_{i=0}^{p} (a_{1,i} + \cdots + a_{m,i}) = 0.$$

Proof. Let system (3) be consistent and let, for instance, q_i be a solution of (3). Then $a_{1,i} = a_{2,i} = \cdots a_{m,i} = 0$, which implies $a_{1,i} + a_{2,i} + \cdots + a_{m,i} = 0$. Therefore (4). Conversly, let (4) hold. Then there is j such that $a_{1,j}+a_{2,j}+\cdots+a_{m,j}=0$. Last equality implies $a_{1,j}=a_{2,j}=\cdots=a_{m,j}=0$ i.e. q_j is a solution of (3). ■

Let us write the system (3) in the form $E_1(x)\wedge\cdots\wedge E_m(x)$.

Definition 3. A formula $x=g(t)$, where $g:Q\rightarrow Q$ is a given function, represents a general solution of the system (3) if and only if

$$(\forall t)(E_1(x)\wedge\cdots\wedge E_m(x))\wedge(\forall x)(E_1(x)\wedge\cdots\wedge E_m(x)\Rightarrow(\exists t)x=g(t)).$$

Definition 4. A formula $x=g(t)$, where $g:Q\rightarrow Q$ is a given function, represents a reproductive general solution of the system (3) if and only if

$$(\forall t)(E_1(x)\wedge\cdots\wedge E_m(x))\wedge(\forall t)(E_1(t)\wedge\cdots\wedge E_m(t))\Rightarrow t=g(t)).$$

Theorem 3 *If system (3) is consistent, then a formula $x=G(t)$ represents a general solution of (3) if $G(t)$ is of the form*

$$\begin{aligned} G(t) \;=\; & \sum_{k=0}^{p}\Big(a^0_{1,i_{k,0}}\cdots a^0_{m,i_{k,0}}\,q_{i_{k,0}}+(a_{1,i_{k,0}}+\cdots+a_{m,i_{k,0}})\,a^0_{1,i_{k,1}}\cdots a^0_{m,i_{k,1}}\,q_{i_{k,1}} \\ & + (a_{1,i_{k,0}}+\cdots+a_{m,i_{k,0}})(a_{1,i_{k,1}}+\cdots+a_{m,i_{k,1}})\,a^0_{1,i_{k,2}}\cdots a^0_{m,i_{k,2}}q_{i_{k,2}} \\ & + \cdots+(a_{1,i_{k,0}}+\cdots+a_{m,i_{k,0}})(a_{1,i_{k,1}}+\cdots+a_{m,i_{k,1}})\cdots \\ & (a_{1,i_{k,p-1}}+\cdots+a_{m,i_{k,p-1}})\,a_{1,i_{k,p}}\cdots a_{m,i_{k,p1}}\,q_{i_{k,p}}\Big)\,t^{q_k}, \end{aligned} \qquad (4)$$

where

$$(i_{k,0},i_{k,1},\ldots,i_{k,p}) \quad \text{are permutations of} \quad \{0,1,\ldots,p\} \qquad (5)$$

and

$$(i_{0,0},i_{1,0},\ldots,i_{p,0}) \quad \text{is a permutation of} \quad \{0,1,\ldots,p\}. \qquad (6)$$

If the condition

$$(i_{0,0},i_{1,0},\ldots,i_{p,0})=(0,1,\ldots,p) \qquad (7)$$

is fulfilled then formula $x=G(t)$ represents reproductive general solution.

Proof. Suppose that $G(t)$ is of the form (5). Let $t=q_k$ for some $k\in\{0,1,\ldots,p\}$ and $(a_{1,i_{k,r}},\ldots,a_{m,i_{k,r}})$ be the first element of the sequence

$$(a_{1,i_{k,0}}+\cdots+a_{m,i_{k,0}}),(a_{1,i_{k,1}}+\cdots+a_{m,i_{k,1}}),\ldots,(a_{1,i_{k,p}}+\cdots+a_{m,i_{k,p}})$$

such that

$$(a_{1,i_{k,0}}+\cdots+a_{m,i_{k,0}})=(a_{1,i_{k,1}}+\cdots+a_{m,i_{k,1}})=\cdots=(a_{1,i_{k,r-1}}+\cdots+a_{m,i_{k,r-1}})=1$$

and $a_{1,i_{k,r}} + \ldots + a_{m,i_{k,r}} = 0$. There is such element because of (4) and (6). Now formula $x = G(t)$ gives

$$\begin{aligned} x &= 0 + (0 + (a_{1,i_{k,0}} + \cdots + a_{m,i_{k,0}}) \cdots a^0_{1,i_{k,r}} \cdots a^0_{m,i_{k,r}} \cdot q_{i_{k,r}} + 0)q_k^{q_k} + 0 \\ &= (1 \cdot q_{i_{k,r}}) \cdot 1 = q_{i_{k,r}}. \end{aligned}$$

Since $a_{1,i_{k,r}} + \ldots + a_{m,i_{k,r}} = 0$ i.e. $a_{1,i_{k,r}} = \ldots = a_{m,i_{k,r}} = 0$, $q_{i_{k,r}}$ is a solution of (3).

Let q_i satisfy (3). Then $a_{1,i} = \cdots = a_{m,i} = 0$. In accordance with (6), there is $s \in \{0,1,\ldots,p\}$ such that $i_{s,0} = i$. If we take $t = q_s$, formula (5) gives

$$\begin{aligned} G(t) &= (a^0_{1,i_{k,0}} a^0_{2,i_{s,0}} \cdots a^0_{m,i_{s,0}} \cdot q_{i_{s,0}} \\ &+ (a_{1,i_{s,0}} + \cdots + a_{m,i_{s,0}}) a^0_{1,i_{k,0}} a^0_{2,i_{s,0}} \cdots a^0_{m,i_{s,0}} + \cdots) q_s^{q_s} \\ &= (1 \cdot q_{i_{s,0}} + 0) \cdot 1 = q_i. \end{aligned}$$

Suppose that the condition (8) is fulfilled. Then

$$a_{j,i_{k,0}} = a_{j,k} \quad (j = 1,\ldots,m;\ k = 0,1,\ldots,p)$$

and

$$q_{i_{k,0}} = q_k \quad (i = 0,1,\ldots,p).$$

Let q_u be a solution of (3). If we take that $t = q_u$ then

$$G(q_u) = 0 + (a^0_{1,u} \cdots a^0_{m,u} q_u + \cdots) q_u^{q_u} + 0.$$

Since q_u is a solution of (3), we have $a_{1,u} = \cdots = a_{m,u} = 0$. Therefore

$$G(q_u) = (1 \cdot q_u + 0) \cdot 1 = q_u. \qquad \blacksquare$$

Example. Let us solve the system of equations

$$ax \cup bx' = 0 \wedge cx \cup dx' = 0 \wedge ex \cup fx' = 0. \tag{8}$$

in Boolean algebra $B_2 = (\{0,1\}, \cap, \cup, ')$ (more facts on Boolean equations can be found in [3]). If we take $x^0 = x'$, $x^1 = x$ and $(\cdot, \vee) = (\cap, \cup)$ we can remark that (9) is a system of Prešić's equations. The consistency condition is $(a \cup c \cup e)(b \cup d \cup f) = 0$. Using formula (5) we get

$$\begin{aligned} g(t) &= (f_1'(0)\, f_2'(0)\, f_3'(0)\, 0 \cup (f_1(0) \cup f_2(0) \cup f_3(0))\, f_1'(1)\, f_2'(1)\, f_3'(1)\, 1) t^0 \\ &\cup ((f_1'(1)\, f_2'(1)\, f_3'(1)\, 1 \cup (f_1(1) \cup f_2(1) \cup f_3(1))\, f_1'(0)\, f_2'(0)\, f_3'(0)\, 0) t^1 \\ &= (b \vee d \vee f)(a \cup c \cup e)' t' \cup (a \cup c \cup e)' t. \end{aligned}$$

Implication $pq = 0 \Rightarrow p'q = q$ and consistency condition $(a \cup c \cup e)(b \cup d \cup f) = 0$ gives $(b \cup d \cup f)(a \cup c \cup e)' = b \cup d \cup f$. In accordance with Theorem 3, formula

$$x = (b \cup d \cup f) t' \cup (a \cup c \cup e)' t$$

represents the reproductive general solution of the system (9).

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