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Thermodynamic Plasma Properties Near the Sheath Edge in Kinetic Tonks-Langmuir Model with Finite Ion Source Temperatures

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Abstract: Modeling plasmas in fluid codes is essentially limited to the region of plasma quasi-neutrality since, due to their high thermodynamic equilibrium, fluid models near the plasma boundaries fail. Recently a concept of the polytropic coefficient function, which is a local quantity (rather than a constant, as usually assumed in plasma physics), has been introduced by Kuhn et al. [Phys. Plasmas 13, 013503 (2006)] . This concept has been already applied to the Tonks-Langmuir discharges in the case of ions created in plasma from a cold ion source, yet, due to the non-reliability of the existing models never to the important case with finite ion sources. Recently a highly reliable solution of the plasma equation with finite temperature in the limit $\varepsilon \equiv \lambda_D/\ell = 0$ (where λ_D is the Debye length and ℓ is a proper characteristic length of the discharge) has been reported by Kos et al. [Phys. Plasmas 16, 093503 (2009)]. Unlike previous Bissell-Johnson models [Phys. Fluids 30, 779 (1987)], the validity of which was limited to the rather narrow ranges of ion source temperatures, with the model by Kos et al. this range is unlimited and solutions are obtained with a high reliability and in a high resolution. Here we employ this model to find relevant plasma parameters at the sheath edge. Special attention is given to the fluid Bohm criterion, which with the ion polytropic coefficient function turns out to be exact. It shows that a kinetic generalization of this criterion might be disregarded for practical purposes.

Keywords: plasma polytropic coefficients, plasma-sheath boundary, fusion applications, integro-differential equations, Bohm criterion

1 Introduction

The Tonks-Langmuir [25] problem with collisionless discharges is regarded as one of the core problems in the area of basic plasma physics and application in space laboratory and fusion plasmas, which fails, however, to be satisfactorily solved even with rather

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simple physical scenarios of interest. This results from the extreme non-linearity of the problem, which we illustrate via our mathematical formulation in the form of a general integro-differential equation

$$\varepsilon^2 n(\Phi)\frac{1}{\Psi^3}\frac{d\Psi}{d\Phi} = 1 - \lambda \int_0^{\Phi} \Psi(\Phi')\mathscr{K}\left(\beta, \tau, \Phi', \Phi\right) d\Phi' \tag{1}$$

where the unknown function to be found is $\Psi(\Phi)$, while other functions., e.g., singular kernel $\mathscr{K}$, and local function $n(\Phi)$ are prescribed in advance, and arbitrary parameters of the problem are β, ε and τ, while the problem eigenvalue is λ. It should be pointed out that such an equation emerges from an elementary physical scenario with numerous approximations and compromises yet, nevertheless, remaining stiff both mathematically and numerically. This requires further assumptions to attack and hopefully solve the equation, as will be done bellow.

In order to introduce the reader with the physical background of Eq. (1), Fig. 1 illustrates the problem with one-dimensional geometry. While details will be elaborated in the next section, we merely note here that the problem consists of finding a potential profile together with ion velocity distribution provided the electron density distribution is known, and the mechanisms of ion production and energy gains and losses are well defined. Schematic potential profile $\Phi(x)$ is shown in the case of a negligible ε. This means that in a very thin sheath region the main potential drop $\Phi_s - \Phi_w$ is located, (where Φ_s - the plasma-sheath potential drop as measured with respect to the center of discharge is the point at which a sudden drop of the electric field $E = -1/\Psi(\Phi)$ is situated, and Φ_w is the wall potential to be found self consistently from the particle flux balance). Due to the symmetry of the problem, only half of the discharge should be considered. In seeking

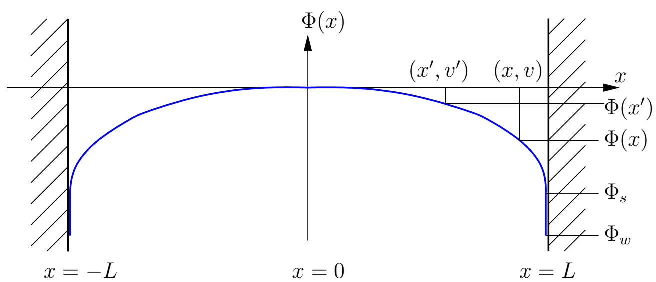

Fig. 1. Schematic diagram of the T&L model in one-dimensional (plane) geometry with potential $\Phi(x)$. The plasma center at $x = 0$, walls at $x = \pm L$. Φ_s is the potential of the sheath edge, Φ_w is the wall potential.

a solution to this type of problem, Tonks and Langmuir (T&L) found that the complete formulation can be split into "plasma approximation," where strict quasi-neutrality is assumed and "sheath approximation", where the electric field dominates. The corresponding two regions of the plasma-wall transition layer are often referred to as "the presheath" and "the Debye sheath" regions. T&L found approximate solutions for these two regions in plane, cylindrical and spherical geometries with the assumption that the ions are generated at rest. This is known as the ”cold” or ”singular” ion source scenario, unlike much more complex the ”warm” or ”regular” ion source case, which is the subject of the present paper. T&L’s “intuitive” approach involves splitting the plasma-sheath equation into two parts, later rendered into a precise mathematical context recognized by Caruso and Cavaliere [6] via employing the boundary layer theory by van Dyke[26]. Proceeding with this approach, Harrison and Thompson (H&T) [9] upgraded the Tonks and Langmuir approximate solution to an exact analytic one. Self [23], however, announced a complete *numerical* solution, i.e., without the quasi-neutrality assumption, but still with a cold ion source.

The first attempt to tackle the plasma solution ($\varepsilon = 0$) with a regular (warm) ion source (the neutral temperature $T_n \neq 0$) has been done by Emmert et al. [8]. They employed an artificial ion source, prepared in advance to yield the Maxwellian ion distribution function. Bissell and Johnson (B&J) [4], on the contrary, *started* from the Maxwellian ion source, and found a numerical solution, unfortunately rather an unreliable one which, in addition, remained limited to a rather narrow range of ion source temperatures. Scheuer and Emmert (S&E) [22] used a better kernel approximation enabling them to find a solution in a wider, yet still limited range which did not cover enough ’warm’ ion sources, which is of high importance to fusion application. Kos et al. [12] and Jelic et al [10], however, have recently managed to employ the exact kernel instead of an approximate one in solving the plasma problem with a ”warm” Maxwellian ion-source without any restriction, for $\varepsilon = 0$ and $\varepsilon \neq 0$ respectively. With regard to the ion sources role, Harrison and Thompson[9] (H&T) defined the problem for a rather general ion source strength profile $S_i \sim n_e$ and solved analytically basic cases $\beta = 0, 1, 2$ with $n_e \sim \exp(\beta\Phi)$, (with Φ the normalized local plasma potential), where case $\beta = 0$ corresponds to the “flat” ion source spatial distribution (e.g., caused by an electron beam or an external laser-caused ionization), $\beta = 1$ corresponds to the single-stage electron-neutral impact ionization and $\beta = 2$ assumes a two-stage ionization mechanism. In addition, a solution has recently been found for warm ion sources [10].

The developments of the computational method [12] and the numerical simulation method [3, 27] today open possibilities to deal with finite ε finite T_n case (see e.g., Refs. [21, 11]). Apparently, a two-scale approach proves to be a less interesting one. By contrast, the analytic determination of the plasma-sheath boundary is regarded of extreme importance in plasma investigations via numerical codes and practical applications. For example, the validity of fusion-relevant codes dealing with the Scrape of Layer, like SOLPS [7] and EDGE2D, is limited to the region bounded by a plasma-sheath surface at which the fluid approach breaks. A well-defined boundary condition requires the employment of the famous Bohm criterion [5], i.e., its generalization [9]. The Bohm criterion is well elaborated in its fluid and hydrodynamic counterparts for $T_n = 0$, yet in the finite ion-source temperature cases it is far from being proven even in the fluid approach. In this paper we show the most advanced formulation of the Bohm criterion in the fluid approach based on new formulation of the ion polytropic coefficient function [15, 14]. Moreover, we demonstrate that our formulation sufficiently defines the plasma-sheath boundary without invoking the kinetic approach at all.

2 Theoretical considerations

The general formulation of the problem as defined by Tonks and Langmuir (T&L) in 1929 [25] for plane-parallel geometry consists in simultaneously solving Boltzmann's equation for the ion VDF, $f_i(x,v)$,

$$v\frac{\partial f_i}{\partial x} - \frac{e}{m_i}\frac{d\Phi}{dx}\frac{\partial f_i}{\partial v} = S_i(x,v) \qquad \text{and} \qquad -\frac{d^2\Phi}{dx^2} = \frac{e}{\varepsilon_0}(n_i - n_e)\ , \tag{2}$$

where the collisional source term $S_i(x,v)$ on the right-hand side is a function describing the relevant microscopic physics involved in the model of interest, with x the Cartesian space coordinate, v the particle velocity, e the positive elementary charge, m_i the ion mass, and $\Phi(x)$ the electrostatic potential at position x), and Poisson's equation for the potential, respectively, where ε_0 is the vacuum dielectric constant, and $n_{i,e}$ are the ion and electron densities, respectively, with additional assumptions and proper boundary conditions. We introduce the normalized quantities of interest as follows:

$$\begin{gathered}\frac{e\Phi}{kT_e} \to \Phi\ , \quad \frac{m_i v^2}{2kT_e} \to v^2\ , \quad \frac{x}{L} \to x\ , \quad \frac{n_{i,e}}{n_{e0}} \to n_{i,e}\ , \quad \frac{T_n}{T_e} \to T_n\ , \\ \frac{T_{i,src}}{T_e} \to T_{i,src}\ , \quad \frac{T_i}{T_e} \to T_i\ , \quad \frac{\sqrt{2}c_{s0} f_i}{n_{e0}} \to f_i\ , \quad S_i L \to S_i\ ,\end{gathered} \tag{3}$$

where $c_{s0} \equiv \sqrt{kT_e/m_i}$ and L is any characteristic system length, (usually, the half-length of the plane-parallel discharge). Eqs. (2) in the normalized forms read:

$$\frac{\partial f_i}{\partial x} - \frac{d\Phi}{dx}\frac{\partial f_i}{(\partial v^2)} = \frac{S_i(x,v)}{v}\ , \qquad \text{and} \qquad -\varepsilon^2\frac{d^2\Phi}{dx^2} = n_i - n_e\ , \tag{4}$$

respectively. Here $\varepsilon \equiv \lambda_D/L$ (with the Debye length $\lambda_D = \sqrt{\varepsilon_0 kT_e/n_{e0}e^2}$ and n_{e0} the electron density at the center of the plasma) is the smallness parameter of the problem. Equation (4) shows that for $\varepsilon \to 0$ the quasi-neutrality condition holds up to the wall where an infinitely thin sheath forms, characterized by infinite electric field. If, on the other hand, "sheath scaling" $x/\lambda_D \to x$ is employed, the sheath becomes infinitely wide.

Assuming that the electron density is Boltzmann-distributed $n_e = \exp(\Phi)$, the procedure described in Ref. [13] leads to the solution in the form:

$$\begin{aligned}&B\int_0^1 dx' \exp[\Phi(x') - \Phi(x)] \exp\left[\frac{1}{2T_n}\{\Phi(x') - \Phi(x)\}\right] \mathrm{K}_0\left\{\frac{1}{2T_n}|\Phi(x') - \Phi(x)|\right\} \\ &= 1 - \varepsilon^2 \exp(-\Phi)\frac{d^2\Phi}{dx^2}\end{aligned}\ , \tag{5}$$

with B emerging from the condition of the charge flux balance in the form [12]:

$$B = \frac{1}{2\pi}\sqrt{\frac{T_e m_i}{T_n m_e}\frac{n_0}{n_{av}}} \exp\left(\frac{e\Phi_w}{kT_e}\right)\ , \tag{6}$$

with Φ_w the wall potential and n_{av} the average ion density.

Note that Eq. (5) depends on configuration space x. In the plasma-sheath problem we assume that the potential profile $\Phi(x)$ is monotonic, so that the inverse function $x(\Phi)$ is monotonic as well, the mathematical rule: $d^2y/dx^2 = -(d^2x/dy^2)/(dx/dy)^3$ holds. Then the elegant form of Eq. (1) is obtained after interchanging the dependent and independent variables $\Psi(\Phi') = dx'/d\Phi' \equiv -1/E$. In fact, this is the Bissell-Jonson approach, which caused theme extreme trouble. Nevertheless, interchanging the dependent and independent variables yields the B&J formula:

$$\frac{1}{B} = \int \Psi(\Phi') \exp\left[\left(1 + \frac{1}{2T_n}\right)(\Phi - \Phi')\right] \mathrm{K}_0 \left(\left|\frac{\Phi - \Phi'}{2T_n}\right|\right) d\Phi' \,. \tag{7}$$

Once a numerical solution of above equation is obtained, it is straightforward (but not easy) to calculate the ion velocity distribution, which in normalized variables in accordance to B&J reads

$$f_i(\Phi(x), v) = B \int_{\Phi'} \Psi(\Phi') \exp(\Phi') \frac{\exp\left[-(v^2 - (\Phi' - \Phi))/T_n\right]}{\sqrt{v^2 - (\Phi' - \Phi)}} d\Phi' \,. \tag{8}$$

Furthermore, all the moments of ion VDF, i.e. the density ($n = \int f(v)dv$), directional velocity ($u = \frac{1}{n}\int f(v)vdv$), and ion temperature $T = \int \frac{1}{n} f(v)(v-u)^2 dv$ and all higher moments like heat flux, energy flux etc., can be found at any location, and of course the quantity $< v^-2 >= \frac{1}{n}\int f(v)dv/v^2$ necessary for the calculation of the H&T plasma-sheath condition.

Finally, once the moments of the velocity distribution function are known, the special quantity of our interest the polytropic coefficient $\gamma_i(x)$ (or equivalently$\gamma_i(\Phi)$) can be found by using the expression:

$$\gamma_i = 1 + \frac{n_i}{T_i}\frac{dT_i}{dn_i} \equiv 1 + \frac{n_i}{T_i}\frac{dT_i/d\Phi}{dn_i/d\Phi} \,. \tag{9}$$

On the other hand, the purpose of our paper is to deal with hydrodynamic properties at the plasma boundary, where a standard procedure of expanding the charge density $n_e - n_i$ in terms of the potential $\Phi(x)$ near the ”infinitely distant” point $x_s/L \to \infty$, $\Phi_s \to 0$, where conditions $(n_e - n_i) \to 0$ and $d\Phi/dx \to 0$ hold. Under these conditions the linearized PoissoN equation takes the form

$$\frac{\varepsilon_0}{2}\left(\frac{d^2\Phi}{dx^2}\right)^2 = \frac{1}{2}\frac{d(n_e - n_i)}{d\Phi}\Phi^2 \,, \tag{10}$$

from where it follows that the condition

$$\frac{d(n_i - n_e)}{d\Phi} \leq 0 \tag{11}$$

must hold near the sheath boundary.

Although it has been argued by Riemann [see e.g., Refs. [16, 17])] that whereas the above expansion is valid in the fluid approach it is inapplicable in the kinetic approach, where another expansion should be applied, for our present purposes it can be sufficiently used as a universal one. We will proceed with the sheath analysis, employing it here further. Then from the Vlasov equations:

$$v\frac{\partial f_{i,e}}{\partial x} \mp \frac{e}{m_{i,e}}\frac{d\Phi}{dx}\frac{\partial f_{i,e}}{\partial v} = 0 \ , \tag{12}$$

or, alternatively, from systems of fluid equations:

$$\begin{aligned} &\frac{\partial(n_{i,e}u_{i,e})}{\partial x} = 0 \ , \\ &m_{i,e}n_{i,e}u_{i,e}\frac{\partial u_{i,e}}{\partial x} + n_{i,e}\frac{d\Phi}{dx} \pm \gamma_{i,e}kT_{i,e}\frac{\partial n_{i,e}}{\partial x} = 0 \ . \end{aligned} \tag{13}$$

assumed to hold in the collisionless sheath region, respective pairs of equalities can be obtained via

$$\begin{aligned} \frac{dn_{i,e}}{d\Phi} &= \pm\frac{e}{m_{i,e}}\int\frac{1}{v}\frac{\partial f_{i,e}}{\partial v}dv \ , \\ \frac{dn_{i,e}}{d\Phi} &= \mp\frac{en_{i,e}}{\gamma_{i,e}kT_{i,e} - m_{i,e}u_{i,e}^2} \ , \end{aligned} \tag{14}$$

which suffices for our present purpose, i.e., for the present modeling of the edge of a strongly localized electric field region. Thus in the most general kinetic case the Harrison and Thompson [9] (H&T) generalization

$$\sum_{i,e}\frac{e}{m_{i,e}}\int\frac{1}{v}\frac{\partial f_{i,e}}{\partial v}dv = \sum_{i,e}\frac{e}{m_{i,e}}\int\frac{f_{i,e}}{v^2}dv \le 0 \tag{15}$$

of the Bohm [5] criterion should be employed, where alternative form with $< v^{-}2 >$ can be obtained after partial integration.

Typically, however, the function of the electron velocity distribution or at least the electron density profile is assumed to be known. Thus treating the ion populations kinetically, the generalized H&T criterion turns somewhat more explicit:

$$m_{i,e}\left(\langle v_i^2\rangle\right)^{-1} \ge en_e\left(\frac{dn_e}{d\Phi}\right)^{-1} \ , \tag{16}$$

where the right-hand side should be calculated either kinetically from the *known* electron velocity distribution function or the electron density profile $n(\Phi)$. Here the term $T_{e,scr} \equiv en_e(dn_e/d\Phi)^{-1}$??is known as the ”screening temperature” [18, 17]. In the case of Maxwellian electrons the screening temperature coincides with the global electron temperature uniformly valid in the whole discharge. Hence, if the ions are modeled in the fluid approach, the criterion (11) takes the form:

$$m_iu_i^2 \ge \gamma_ekT_e + \gamma_ikT_i = c_s^2 \ , \tag{17}$$

where c_s^2 has the same structure as "classical" ion sound velocity. In normalized variables so we may write:

$$u_i \geq \sqrt{1+\gamma_i T_i}\ . \tag{18}$$

However, the *essential* novelty of the present work is that γ may *not* be considered as a constant (taking values 1, 5/3, or 3 for the isothermal, adiabatic flow with isotropic pressure and 3 for the one-dimensional adiabatic flow), as presented in any classic textbook on plasma physics, but that it is a constant which, moreover, assures that the *exact equality* in the Bohm condition (18) takes place, rather than inequality (known as the "marginal Bohm condition"). This means that the plasma sheath boundary, being a point of the electric field singularity, is the "sonic" surface (Mach number equal to unity) according to Stengeby and Allen's hypothesis made in fluid theory [24] and Allen's hypothesis argued for the kinetic model. [2], both made on the basis of comparing the dispersion relations in the limit of vanishing phase velocity $\omega/k \to 0$ and small wave vector $k\lambda_D \ll 1$ with the Bohm criteria in both the fluid and kinetic approaches.

3 Results

Our interest here is Eq. (6) in the limit of vanishing ε:

$$\frac{1}{B} = \int_0^1 dx' \exp\left[\left(\beta+\frac{1}{2T_n}\right)\Phi(x') - \left(1+\frac{1}{2T_n}\right)\Phi(x)\right] \mathrm{K}_0\left(\frac{1}{2T_n}\left|\Phi(x')-\Phi(x)\right|\right)\ , \tag{19}$$

where parameter β characterizes the ionization mechanism distribution [10] and takes e.g.,

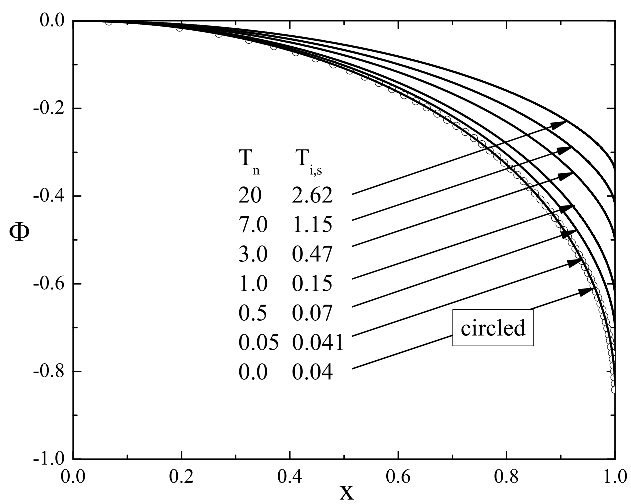

Fig. 2. Potential profiles obtained for different ion source (neutral gas) temperatures. The ion temperatures at the plasma sheath edge are indicated in parallel.

value $\beta = 0$ for uniformly distributed ion source and $\beta = 1$ for the ion source proportional to the electron density (see e.g. Jelic et al. 2009 [10]). Our code employs the piecewise

Lagrangian interpolation of order 2 or 3 in the areas of mild $\Phi(x)$ gradients, supporting iterations with high accuracy within wide ion-source temperature ranges, especially in the limit $T_n \to 0$, which is sensitive to instabilities due to the prolonged integration intervals caused by $1/T_n$ singularity [see Fig. 2 presents only the potential profiles obtained for $\beta = 1$ (while results obtained for $\beta = 0$ are omitted, yet will be discussed at a later point) in a wide range of ion source temperatures. With any of particular T_n, potential profiles start at $\Phi = 0$ and end at Φ_s independently of value of β. With exception of these end points the pairs of curves $\beta = 1$ of $\beta = 0$ differ in all other points, reflecting the well-known property of the T&L solution that the plasma solution is invariant with respect to the potential, but is *not* invariant with respect to the spatial coordinate. An important fact to be noted is that the numerically obtained curves for $T_n = 0.01$ via our method excellent approximate the exact analytic curves for $T_n = 0$, so that for the sake of practicality they might be mutually substituted.

For zero ion-source temperature $T_n = 0$, the exact solution holds in the range from 0 to $\Phi_s = -0.85403...$ (maximum of the Dawson function [1] $F_D(\sqrt{z})$), which gives the

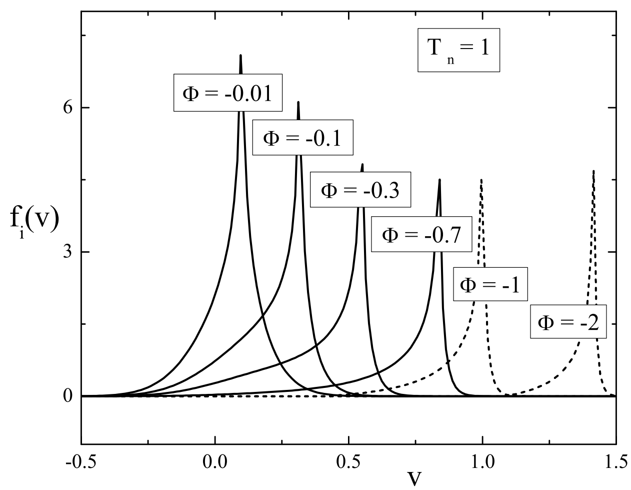

Fig. 3. Ion velocity distribution functions at several positions in the discharge (dashed lines indicate VDF's in the sheath region).

system length of the discharge (see e.g., Riemann[19]). For a comparison with normalized system length $L = 1$, inverse function $\Phi(z)$ can be numerically solved by finding the root of $z - x(\Phi)/L_0 = 0$. Although Eq. (19) can be evaluated to arbitrary precision to simulate high grading near the sheath edge used in the warm case, the potential curve is positioned at the following discrete positions

$$x_i = \left(1 - \left(1 - \frac{i}{np-1}\right)^{\lambda_2}\right)^{\lambda_1}, \quad i = 0, 1, \ldots, np-1\,, \tag{20}$$

where number of points np and grading at endpoints λ_1 and λ_2 should be similar to those used in warm case $T_n > 0$. Our curves in Fig. 2 are all normalized to the unit length to enable a comparison of the shapes. The important fact is that for any temperature there is a value of the plasma potential $\Phi_s(T_n)$ for which the electric field becomes infinite. These points of breaking quasi-neutrality are identified as the plasma sheath boundaries (as functions of the ion source temperatures). According to the detailed results in Jelic et al. [10], the potential drop $|\Phi_s(T_n)|$ decreases with increased temperature T_s.

While VDF for the cold ion-source is the Dirac δ-function, for the finite ion-source temperatures $T_n > 0$ a variety of VDFs are possible. For the γ-processing we used the Maxwellian ion-source, as results were readily available with various grid setups, so we could also test the grid invariance. The calculation of the potential profiles for the whole temperature range and different grids took more than 700000 processor hours. Note that in Fig. 2 the ions e.g., "recycled" from very hot neutrals in fusion plasmas, are much "colder" than the sources (see the temperature profiles in the figures bellow).

With known potential profiles the ion velocity distributions at any point of the discharge can be obtained via e.g., the trajectory method (see e.g., Kos et al [12]). For the sake of curiosity, we illustrate velocity distributions at several places in the discharge both in plasma (solid lines) and the sheath (dashed lines) in Fig. 3 for a particular source temperature $T_n =$ 1. It should be noted that calculating the ion velocity distributions can be facilitated through a simple shift in the energy coordinate. In fact, calculation of ion velocity distributions at an arbitrary high number of the positions is considerably less computationally demanding and considerably less expensive than calculating the profiles, which requires huge CPU resources. Finding the fluid quantities profiles is the next task to be done towards obtaining

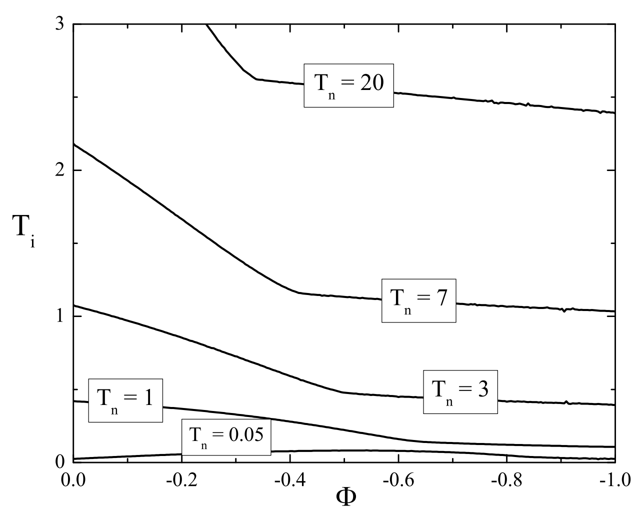

Fig. 4. Ion-temperature profiles for various ion-source temperatures.

the complete information necessary for closing the set of the data basis required for any practical purposes. We do not show the ion density profiles since, by the definition of the problem, they are identical to the electron density profiles. For illustration, Fig. 4 shows the temperature profiles for several ion-source temperatures, wherefrom it is clear that the final ion temperatures are considerably bellow the ion source temperatures. The knees of the temperature profiles correspond to the points of the plasma-sheath boundaries (there is an essential difference in the quantity of profiles with small temperatures, as discussed in detail in Ref. [15]). The essential results for this paper are illustrated in Fig. 5, where we

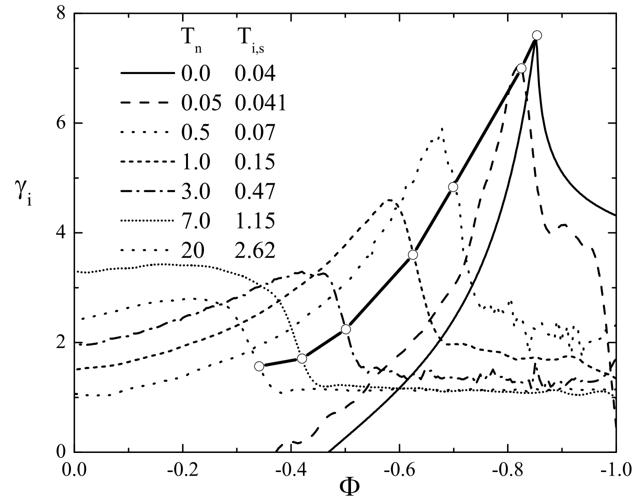

Fig. 5. Ion polytropic coefficient function profiles for different ion-source temperatures. Small circles indicate the plasma boundaries ($\Phi_s=$ -0.854, -0.825, -0.699, -0.6249, -0.501, -0.42, -0.341) for each particular ion-source temperature ($T_n=$ 0, 0.05, 0.5, 1, 3, 7, 20).

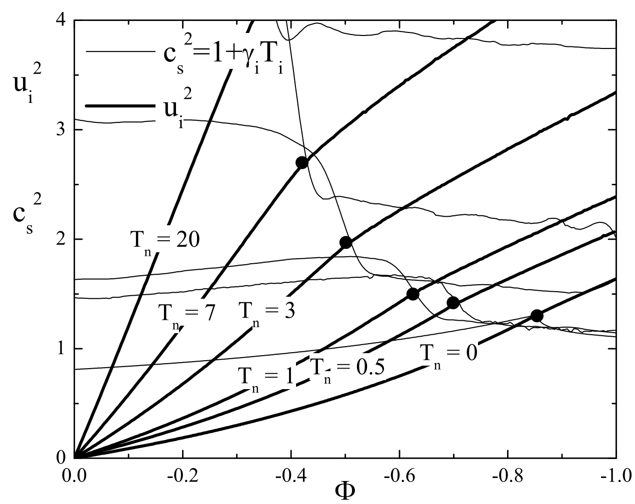

Fig. 6. Profiles of the square of the ion directional velocity (u_i^2 - thick lines) and the square of the ion-sound velocity $c_s^2 = 1+\gamma_i T_i$

show several profiles of the ion polytropic coefficient functions for several ion sources. The limiting case obtained in our previous works for $T_n = 0$ is marked by a thick line. Sharp characteristic peak appears in this case due to the inflection point of the temperature profile for cold ion source for $T_n \simeq 0$ as discussed in Refs. [15, 20]. It is important to remember the shape and the level of the bold curve joining the values of γ_i's in the large range of ion source temperatures. Obviously, it is difficult to talk about any particular choice in which γ_i takes particular values of e.g., either 1 or 3. The values characterizing the plasma-sheath positions are marked via values of Φ_s for each particular T_n used in this example.

The above-mentioned results seem to be at least qualitatively known from the cited literature. The key result of the present work, however, is illustrated in Fig. 6 as follows. Profiles of the square of the ion directional velocity $u_i^2(\Phi)$ are presented in Fig. 6 (thick lines) together with the squares of the ion-sound velocity $c_s^2(\Phi) = 1 + \gamma_i T_i$ profiles (thin lines). For any particular temperature T_n the points of $u_i^2(\Phi_s)$ are marked with full circles. It is obvious that the circles coincide with the intersection points ($u_i^2 = c_s^2$). This statement is valid with certain accuracy caused by ad hoc smoothing derivatives $dT_i/d\Phi$ necessary for calculation of smooth γ_i via formula Eq. (9). Fig. 6 appears to be a rather spectacular one since the involvement of the local polytropic coefficient function into the plasma-sheath theory as a substantially relevant quantity is a considerable improvement of the formula for the Bohm criterion not only in the sense of quantitative determination of the Bohm velocity, but also substantially from the point of view of the basic principles of plasma physics.

4 Conclusion

The authors take the freedom to claim that the results presented in the present work are both surprising and superior to the classic results as presented in standard course-books on basic plasma physics or in highly refined available plasma-sheath theories. Firstly, we confirm that the Bohm criterion with the marginal (equality) sign holds at the plasma boundary provided the exact *local* value of ion polytropic coefficient function is known. Secondly, it turns out that once the local polytropic coefficient function is known, *no* kinetic criterion such as Harrison and Thompson's is needed, since these kinetic formulations have no measurable physically relevant plasma parameter. Finally, since our criterion with local polytropic coefficient appears to be correct, it turns out that any fluid formulation known from published works with constant γ is an *oversimplification* of the Nature in comparison with the present model, despite the seeming structural similarities in the formulas. The task remaining to be done in future is to possibly show via *analytic* arguments that our formula is a universal one irrespective of the details of the discharge such as the shape of the ion velocity distribution source, ionization mechanism, the geometry of the discharge etc.

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