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Forced Oscillations of a Single Degree of Freedom System with Fractional Dissipation

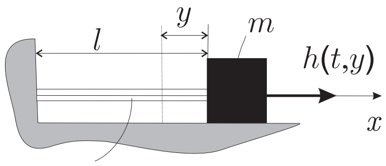
T. M. Atanacković, D. Ć. Dolićanin, S. Pilipović

Abstract: We study motion of a single degree of freedom mechanical system consisting of a visco-elastic rod of finite length with concentrated mass at the free end. If the deformation of the rod is approximated so that rod is considered in a state of quasi-static deformation or, equivalently, if it is assumed that the rod is light (density is equal to zero) we show that the several known oscillation equations can be derived.

Keywords: Fractional derivative, distributed-order fractional derivative.

1 Introduction

Fractional order viscoelasticity has been treated recently in many publications [12], [13], [11], [2], [8],[9]. In this work we consider a viscoelastic rod of finite length, fixed at one end and with a mass attached at the other end. The mass is restricted to move along a straight line coinciding with the rods axis (see Figure)



viscoelastic rod

We consider motion of the mass from the position $y = 0$ in which the rod is undeformd, i.e., it has its natural length l. The function h representing the outer forces is assumed to be known function of time t and displacement of the point at which the mass is conected. The problem of determining the motion of an oscillator presented in Fig. 1 was treated earlier in [4]. The deformation of the rod is described by the following system of equations

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T. M. Atanacković is with the Faculty of Technical Sciences, Institute of Mechanics, University of Novi Sad, Novi Sad, Serbia; D. Ć. Dolićanin is with the Faculty of Technical Sciences, University of Pristina - Kosovska Mitrovica, Serbia; S. Pilipović is with the Faculty of Science, Department of Mathematics, University of Novi Sad, Novi Sad, Serbia.

$$
\begin{aligned}
&\frac{\partial}{\partial x}\sigma(x,t)=\rho\frac{\partial^2}{\partial t^2}u(x,t),\\
&\int_0^1\phi_1(\alpha)\,{}_0D_t^\alpha\sigma(x,t)\,\mathrm{d}\alpha=E\int_0^1\phi_2(\alpha)\,{}_0D_t^\alpha\mathscr{E}(x,t)\,\mathrm{d}\alpha,\\
&\mathscr{E}(x,t)=\frac{\partial}{\partial x}u(x,t),\ \ x\in[0,L],\ t>0.
\end{aligned}
\tag{1}
$$

Here ρ, σ, u, E, and $\mathscr{E}$ denote density, Cauchy stress, displacement of an arbitrary point of the rod positioned at the distance x from the left end of the rod in the undeformed state, generalized modulus of elasticity, and strain measure of a material at a point positioned at x and at a time t, respectively. Also in (1) we use ${}_0D_t^\alpha\sigma(x,t)$ to denote the left Riemann-Liouville fractional derivative with respect to time. Note that the displacement of the end point of the rod at which the mass is attached is $y(t)=u(l,t)$. Also, in $(1)_2$ ϕ_1 and ϕ_2 denote specified constitutive functions or distributions. There are number of ways how one can specify ϕ_1 and ϕ_2 (see [10]). For example, $\phi_1=\delta$ and $\phi_2=\delta$, with $E>0$ and δ being Dirac distribution, leads to the Hooke Law. Another choice is (see [17])

$$
\phi_1(\alpha)=a^\alpha,\ \ \phi_2(\alpha)=b^\alpha,\ \ \alpha\in(0,1),\ a\leq b,
\tag{2}
$$

where the restriction $a\leq b$ follows from the Second Law of Thermodynamics (see for example [3]). If $a=b$ then $(1)_2$ reduces, again, to the Hooke's Law. Recall, the left Riemann-Liouville fractional derivative of a function $y\in AC([0,T])$, for every $T>0$, of the order $\alpha\in[0,1)$, is defined as

$$
{}_0D_t^\alpha y(t):=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\frac{y(\tau)}{(t-\tau)^\alpha}\mathrm{d}\tau,\ \ t>0,
$$

where Γ is the Euler gamma function and $AC([0,T])$ denotes the space of absolutely continuous functions (see [15]). In the case when ϕ_1 and ϕ_2 are distributions, we assume that ϕ_1 and ϕ_2 are compactly supported by $[0,1]$ ($\phi_1,\phi_2\in\mathscr{E}'(\mathbb{R})$, $\sup\phi_1,\sup\phi_2\subset[0,1]$). In this case integrals in $(1)_2$ are defined as

$$
\left\langle\int_{\sup\phi}\phi(\alpha)\,{}_0D_t^\alpha h(t)\,\mathrm{d}\alpha,\varphi(t)\right\rangle:=\langle\phi(\alpha),\langle{}_0D_t^\alpha h(t),\varphi(t)\rangle\rangle,\ \ \varphi\in\mathscr{D}(\mathbb{R}).
$$

For details see [6]. Recall, $\mathscr{D}'_+(\mathbb{R})$ denotes the space of distributions supported by $[0,\infty)$ and $\langle h(t),\varphi(t)\rangle$ denotes the action of a distribution $h\in\mathscr{D}'_+(\mathbb{R})$ on a test function $\varphi\in\mathscr{D}(\mathbb{R})$ (see [16]). The constitutive equations of type $(1)_2$ were used earlier in [2], [5], [7] and [10].

The boundary conditions corresponding to a rod shown in Figure read

$$
u(x,0)=0,\ \ \frac{\partial}{\partial t}u(x,0)=0,\ \ \sigma(x,0)=0,\ \ \mathscr{E}(x,0)=0,\ \ x\in[0,l],
\tag{3}
$$

$$
u(0,t)=0,\ \ \sigma(l,t)=\Sigma(t,u(l,t)),\ \ t\in\mathbb{R},
\tag{4}
$$

where $\Sigma(t,u(l,t)) = h(t,u(l,t)) - \frac{m}{A}\frac{\partial^2 u(l,t)}{\partial t^2}$, and m is the mass of the body attached to the rod end and A is the coross-sectional area of the rod. Introducing the quantities

$$\begin{aligned}
\bar{x} &= \frac{x}{l},\ \bar{t} = \frac{t}{l\sqrt{\frac{\rho}{E}}},\ \bar{u} = \frac{u}{l},\ \bar{\sigma} = \frac{\sigma}{E},\ \bar{\Sigma} = \frac{\Sigma}{l},\ \bar{\phi}_1 = \frac{\phi_1}{\left(l\sqrt{\frac{\rho}{E}}\right)^{\alpha}},\ \bar{\phi}_2 = \frac{\phi_2}{\left(l\sqrt{\frac{\rho}{E}}\right)^{\alpha}},\\
\overline{m} &= \frac{m}{Al},
\end{aligned}$$

and using the fact that the fractional derivative transforms as

$${}_0D_{\bar{t}}^{\alpha}u(\bar{t}) = \left(l\sqrt{\frac{\rho}{E}}\right)^{\alpha}{}_0D_t^{\alpha}u(t),$$

we obtain, after omitting bar over dimensionless quantities, the following system

$$\begin{aligned}
&\frac{\partial}{\partial x}\sigma(x,t) = \frac{\partial^2}{\partial t^2}u(x,t),\\
&\int_0^1 \phi_1(\alpha)\,{}_0D_t^{\alpha}\sigma(x,t)\,\mathrm{d}\alpha = \int_0^1 \phi_2(\alpha)\,{}_0D_t^{\alpha}\mathscr{E}(x,t)\,\mathrm{d}\alpha,\\
&\mathscr{E}(x,t) = \frac{\partial}{\partial x}u(x,t),\ \ x\in[0,1],\ t>0.
\end{aligned}\tag{5}$$

System (5) is subject to initial

$$u(x,0)=0,\ \ \frac{\partial}{\partial t}u(x,0)=0,\ \ \sigma(x,0)=0,\ \ \mathscr{E}(x,0)=0,\ \ x\in[0,1],\tag{6}$$

and boundary conditions

$$u(0,t)=0,\ \ \sigma(1,t)=\Sigma(t,u(1,t))=h(t,u(1,t))-m\frac{\partial^2 u(1,t)}{\partial t^2},\ \ t\in\mathbb{R}.\tag{7}$$

System (5),(6),(7) describes the motion of an viscoelastic rod with distributed order fractional constitutive equation, with a concentrated mass attached to its end. With $m = 0$ the system (5),(6),(7) was treated in ([17]),([18]). It was shown there that quasi-static solution of this system, i.e., a solution obtained when $\rho = 0$ is inserted in the first equation of (1), approximates well the solution for large times. This motivates us to make the folowing simplification: suppose that the displacement of any point of the rod is a linear function of the displacement of the right end point of the rod, i.e., $u = xu(1,t) = xy(t)$, where y is unknown function. Then the strain $\mathscr{E}(x,t) = \partial u(x,t)/\partial x = y(t)$ is independent of x. Therefore

$$\int_0^1 \phi_1(\alpha)\,{}_0D_t^{\alpha}\sigma(x,t)\,\mathrm{d}\alpha = \int_0^1 \phi_2(\alpha)\,{}_0D_t^{\alpha}y(t)\,\mathrm{d}\alpha,\tag{8}$$

so that $\sigma(x,t) = \sigma(t)$.

Using the fact that $\sigma(x,t) = \sigma(t)$, the equation of motion $(5)_1$ becomes

$$\frac{\partial^2}{\partial t^2}u(x,t) = 0.\tag{9}$$

Thus, $(5)_1$ is satisfied only for quasi-static processes. Equation (7) leads to

$$my^{(2)}(t) = h(t, y(t)) - \sigma(t), \tag{10}$$

and (see (8))

$$\int_0^1 \phi_1(\alpha)\, {}_0D_t^{\alpha}\sigma(t)\, \mathrm{d}\alpha = \int_0^1 \phi_2(\alpha)\, {}_0D_t^{\alpha} y(t)\, \mathrm{d}\alpha, \tag{11}$$

From system (10),(11) we can obtain various single degree of freedom fractional oscillators studied before. We list some of them:

1. Standard linear fractionaly damped oscillator:

   We take $\phi_1(\alpha) = \delta(\alpha)$, $\phi_2(\alpha) = b\, \delta(\alpha - 1)$, $h(t, y(t)) = -\omega^2 y(t)$, $b = const.$, $\omega^2 = const$. Then (10),(11) lead to

$$my^{(2)}(t) + b\, y^{(1)}(t) + \omega^2 y(t) = 0. \tag{12}$$

2. Linear fractionally damped, forced oscillator (see [1], [20], [21]):

   We take $\phi_1(\alpha) = \delta(\alpha)$, $\phi_2(\alpha) = b\, \delta(\alpha - \beta)$, $h(t, y(t)) = -\omega^2 y(t) + h_0(t)$, $b = const.$, $\omega^2 = const.$, $0 < \beta < 1$. In this case system (10),(11) becomes

$$my^{(2)}(t) + b\, {}_0D_t^{\beta} y(t) + \omega^2 y(t) = h_0(t). \tag{13}$$

3. Fractionally damped, Duffing oscillator:

   Let $\phi_1(\alpha) = \delta(\alpha), \phi_2(\alpha) = b\,\delta(\alpha - \beta), h(t, y(t)) = -\omega^2 y(t) + cy^3(t), b > 0$, $c > 0$. Then we have

$$my^{(2)}(t) + b\, {}_0D_t^{\alpha} y(t) + \omega^2 y(t) + cy^3(t) = 0, \tag{14}$$

   which is fractionally damped Duffing oscillator.

4. Linear oscillator with distributed order fractional damping

   In this case we take $\phi_1(\alpha) = a^{\alpha}, \phi_2(\alpha) = b^{\alpha}, h(t, y(t)) = -\omega^2 y(t) - h_0(t)$ where $a < b$ as a consequence of Second law of thermodynamics and $h_0(t)$ is a given function. Then

$$\begin{aligned} my^{(2)}(t) + \omega^2 y(t) + \sigma(t) &= h_0(t), \\ \int_0^1 a^{\alpha}\, {}_0D_t^{\alpha}\sigma(t)\, \mathrm{d}\alpha &= \int_0^1 b^{\alpha}\, {}_0D_t^{\alpha} y(t)\, \mathrm{d}\alpha. \end{aligned} \tag{15}$$

   The system (15) was treated in [4].

2 Solution of (13)

First we present a solution to (13) for a specific choice of parameters and forcing function. Thus, we assume: $m = 1, h_0 = f \sin \Omega t, f = const.$ and the following intial condtions

$$y(0) = y_0, \quad \frac{dy}{dt}(0) = v_0. \tag{16}$$

Thus, the problem becomes: solve

$$y^{(2)}(t) + b\, {}_0D_t^{\alpha} y(t) + \omega^2 y(t) = f \sin \Omega t, \tag{17}$$

subjected to (16).

In order that fractional derivative is well defined we assume $y \in AC^1(0,T)$. By applying the Laplace transform, $\mathscr{L}(y)(s) = Y(s) = \int_0^\infty \exp(-ts)y(t)dt$ to (17) we obtain

$$\begin{aligned} Y(s) &= \frac{sy_0 + v_0}{s^2 + bs^\alpha + \omega^2} + \frac{b\left[{}_0D_t^{\alpha-1} y\right]_{t=0}}{s^2 + bs^\alpha + \omega^2} \\ &\quad + h_0 \frac{\Omega}{(s^2 + \Omega^2)(s^2 + bs^\alpha + \omega^2)}. \end{aligned}$$

where we used:

$$\mathscr{L}({}_0D_t^\alpha y)(s) = s^\alpha Y(s) - \left[{}_0D_t^{\alpha-1} y\right]_{t=0},$$

and

$$\mathscr{L}\left(y^{(2)}\right)(s) = s^2 Y(s) - sy(0) - y^{(1)}(0) = s^2 Y(s) - sy_0 - v_0.$$

Next we determine $\left[{}_0D_t^{\alpha-1} y\right]_{t=0}$. Since $y(0) = y_0$ is finite we have

$$\lim_{t\to 0}\left({}_0D_t^{\alpha-1} y\right) = \lim_{t\to 0}\left[\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{y(\tau)}{(t-\tau)^{\alpha-1}} d\tau\right] = 0. \tag{18}$$

Therefore from (18) we obtain

$$Y(s) = \frac{sy_0 + v_0}{s^2 + bs^\alpha + \omega^2} + h_0 \frac{\Omega}{(s^2 + \Omega^2)(s^2 + bs^\alpha + \omega^2)}, \quad 0 \le \alpha < 1. \tag{19}$$

To find the inversion of (19), we follow the procedure of [1] and use a result of Prüss [19], Corollary 2.5.2. Let us recall this result in the scalar version:

Let $q : \{s; \Re s > 0\} \to \mathbb{C}$ *be holomorphic. If there exists* $M > 0$ *such that* $||sq(s)||_\infty < M$ *and* $||s^2 q'(s)||_\infty < M$, *then there exists a continuous bounded function* f *on* $(0,\infty)$ *such that* $q(s) = \mathscr{L}(f(t))(s), \Re s > 0$.

If we consider

$$Y(s) = \frac{sy_0 + v}{s^2 + bs^\alpha + \omega^2},$$

we first find $s_0 > 0$ so that $|s^2 + bs^\alpha + \omega^2| > 0, \Re s > s_0$. Then, putting $q(s) = Y(s+s_0)$ we simply see that q satisfies the assumptions of the above assertion and since $\mathscr{L}(e^{s_0 t} f(t))(s) = \mathscr{L}(f(t))(s+s_0)$, we obtain that $Y(s), \Re s > s_0$ is the Laplace transform of a continuous function on $(0,\infty)$ so that $|F(t)| \le Ce^{s_0 t}, t > 0$, for some $C > 0$.

Having this in mind, we calculate below formally the expansions into series of functions and find the inverse Laplace transforms not proving for any specific case the convergence and the existence of the Laplace transform. We know that this formal calculus is legitimate and here, up to the end of this section, we just present formal results. (Series which will appear below converge for $\Re s$ enough large.)

Thus, we write

$$\begin{aligned}
\frac{sy_0 + v_0}{s^2 + bs^\alpha + \overline{\omega}^2} &= \frac{sy_0}{s^2 + bs^\alpha + \overline{\omega}^2} + \frac{v_0}{s^2 + bs^\alpha + \overline{\omega}^2} \\
&= y_0 \frac{1}{s} \frac{1}{1 + \frac{b}{s^2}\left(s^\alpha + \frac{\omega^2}{b}\right)} + v_0 \frac{1}{s^2} \frac{1}{1 + \frac{b}{s^2}\left(s^\alpha + \frac{\omega^2}{b}\right)} \\
&= y_0 \sum_{k=0}^{\infty} \frac{(-1)^k b^k}{s^{2k+1}} \left(s^\alpha + \frac{\omega^2}{b}\right)^k + v_0 \sum_{k=0}^{\infty} \frac{(-1)^k b^k}{s^{2k+2}} \left(s^\alpha + \frac{\omega^2}{b}\right)^k \\
&= y_0 \sum_{k=0}^{\infty} (-1)^k \sum_{j=0}^{k} \binom{k}{j} \frac{b^j \omega^{2(k-j)}}{s^{2k+1-\alpha j}} \\
&\quad + v_0 \sum_{k=0}^{\infty} (-1)^k \sum_{j=0}^{k} \binom{k}{j} \frac{b^j \omega^{2(k-j)}}{s^{2(k+1)-\alpha j}}.
\end{aligned} \tag{20}$$

The last term in (19) can be transformed as

$$\begin{aligned}
& h_0 \frac{\Omega}{(s^2 + \Omega^2)(s^2 + bs^\alpha + \omega^2)} \\
&= h_0 \Omega \frac{1}{s^2} \frac{1}{1 + \left(\frac{\Omega}{s}\right)^2} \frac{1}{(s^2 + bs^\alpha + \omega^2)} \\
&= h_0 \Omega \sum_{k=0}^{\infty} (-1)^k \frac{1}{s^2} \left(\frac{\Omega}{s}\right)^{2k} \sum_{k=0}^{\infty} (-1)^k \sum_{j=0}^{k} \binom{k}{j} \frac{b^j \omega^{2(k-j)}}{s^{2(k+1)-\alpha j}} \\
&= h_0 \Omega \sum_{n=0}^{\infty} (-1)^n \sum_{i=0}^{n} \sum_{j=0}^{n-i} \binom{n-i}{j} \frac{b^j \Omega^{2i} \omega^{2(n-i-j)}}{s^{2(n+2)-\alpha j}}
\end{aligned}$$

Therefore (19) may be written as

$$\begin{aligned}
Y(s) &= y_0 \sum_{k=0}^{\infty} (-1)^k \sum_{j=0}^{k} \binom{k}{j} \frac{\mu^j \overline{\omega}^{2(k-j)}}{s^{2k+1-\alpha j}} \\
&\quad + v_0 \sum_{k=0}^{\infty} (-1)^k \sum_{j=0}^{k} \binom{k}{j} \frac{\mu^j \overline{\omega}^{2(k-j)}}{s^{2(k+1)-\alpha j}}
\end{aligned}$$

$$+h_0\Omega\sum_{n=0}^{\infty}(-1)^n\sum_{i=0}^{n}\sum_{j=0}^{n-i}\left(\begin{array}{c}n-i\\j\end{array}\right)\frac{\mu^j\Omega^{2i}\omega^{2(n-i-j)}}{s^{2(k+2)-\alpha j}},\quad 0\leq\alpha<1$$

It could be shown that all series converge, and that the inversion may be performed term by term so that

$$\begin{aligned}y(t) &= y_0\sum_{k=0}^{\infty}(-1)^k\sum_{j=0}^{k}\left(\begin{array}{c}k\\j\end{array}\right)\frac{\mu^j\omega^{2(k-j)}t^{2k-\alpha j}}{\Gamma[2k+1-\alpha j]}\\ &+v_0\sum_{k=0}^{\infty}(-1)^k\sum_{j=0}^{k}\left(\begin{array}{c}k\\j\end{array}\right)\frac{\mu^j\omega^{2(k-j)}t^{2k+1-\alpha j}}{\Gamma[2(k+1)-\alpha j]}\\ &+h_0\Omega\sum_{n=0}^{\infty}(-1)^n\sum_{i=0}^{n}\sum_{j=0}^{n-i}\left(\begin{array}{c}n-i\\j\end{array}\right)\frac{\mu^j\Omega^{2i}\omega^{2(n-i-j)}t^{2n+3-\alpha j}}{\Gamma[2(n+2)-\alpha j]}\\ 0 &\leq \alpha<1,\end{aligned}\tag{21}$$

is a solution to (13),(16) with $h_0(t)=f\sin\Omega t$.

The results (21) may be generalized for the case of arbitrary forcing term. For example, if we consider

$$y^{(2)}(t)+b\,{}_0D_t^{\alpha}y+\omega^2y(t)=h(t).\tag{22}$$

subject to

$$y(0)=y_0,\quad y^{(1)}(0)=v_0.\tag{23}$$

with $h(t)$ arbitrary function we obtain

$$Y(s)=\frac{sy_0+v_0}{s^2+bs^\alpha+\omega^2}+\frac{H(s)}{s^2+bs^\alpha+\omega^2},\quad 0\leq\alpha<1\tag{24}$$

and

$$Y(s)=\frac{v_0}{s^2+s^\alpha+\omega^2}+\frac{H(s)}{s^2+s^\alpha+\omega^2},\quad 1\leq\alpha<2\tag{25}$$

In determining the inversion of (24),(25) we will use the series expanssion of the term $\frac{1}{s^2+s^\alpha+\omega^2}$. Namely since

$$\frac{1}{s^2+bs^\alpha+\omega^2}=\sum_{k=0}^{\infty}(-1)^k\sum_{j=0}^{k}\left(\begin{array}{c}k\\j\end{array}\right)\frac{b^j\omega^{2(k-j)}}{s^{2(k+1)-\alpha j}}.$$

we get, after the use of convolution theorem

$$\begin{aligned}y(t) &= f_0\sum_{k=0}^{\infty}(-1)^k\sum_{j=0}^{k}\left(\begin{array}{c}k\\j\end{array}\right)\frac{b^j\omega^{2(k-j)}t^{2k-\alpha j}}{\Gamma[2k+1-\alpha j]}\\ &+v_0\sum_{k=0}^{\infty}(-1)^k\sum_{j=0}^{k}\left(\begin{array}{c}k\\j\end{array}\right)\frac{b^j\omega^{2(k-j)}t^{2k+1-\alpha j}}{\Gamma[2(k+1)-\alpha j]}\\ &+\int_0^t h(t-\tau)\sum_{k=0}^{\infty}(-1)^k\sum_{j=0}^{k}\left(\begin{array}{c}k\\j\end{array}\right)\frac{b^j\omega^{2(k-j)}\tau^{2k+1-\alpha j}}{\Gamma[2(k+1)-\alpha j]}d\tau.\\ 0 &\leq \alpha<1\end{aligned}\tag{26}$$

For example, if we take $0 \leq \alpha < 1, y_0 = v_0 = 0, H(t) = \delta(t)$ with $\delta(t)$ being the Dirac distribution, the solution (26) becomes

$$f(t) = \sum_{k=0}^{\infty} (-1)^k \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} \frac{b^j \omega^{2(k-j)} t^{2k+1-\alpha j}}{\Gamma[2(k+1) - \alpha j]}. \tag{27}$$

This specific example was treated in [24] p. 5031 by iteration method.

3 Some properties of the oscillator without the forcing term

We consider (12) with (16). The total mechanical energy of the oscillator in this case reads

$$E = \frac{1}{2}\left[m\left(y^{(1)}(t)\right)^2 + \omega^2 (y(t))^2\right]. \tag{28}$$

We have the following Lemma:

Lemma 1 *If $b > 0, 0 < \alpha < 1$ and $v_0 = 0$ the then the total energy of the system described by (12) with (16) is nonincreasing.*

Proof. Multiplying (12) with $y^{(1)}(t)$ and integrating, we obtain

$$E(t) - E(0) = -b \int_0^t {}_0D_\tau^\alpha y(\tau)(\tau) y^{(1)}(\tau) d\tau. \tag{29}$$

Let ${}_0D_t^\alpha y(t) = u(t)$. Then, the integral on the right hand side becomes

$$\int_0^t {}_0D_\tau^\alpha y(\tau) y^{(1)}(\tau) d\tau = \int_0^t u(\tau)\ {}_0D_\tau^\beta u(\tau)(\tau) d\tau, \quad \beta = 1 - \alpha > 0.$$

From [22] (see also [23]) we have that

$$\int_0^t u(\tau)\ {}_0D_\tau^\beta u(\tau)(\tau) d\tau \geq 0$$

so that

$$E(t) - E(0) = -b \int_0^t {}_0D_\tau^\alpha y(\tau)(\tau) y^{(1)}(\tau) d\tau \geq 0$$

and the result follows. ■

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