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Introducing Affine Invariance to IFS

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Abstract: The original definition of the IFS with affine contractive mappings is an important and handy tool for constructive approach to fractal sets. But, in spite of clear definition, the concept of IFS does not allow many possibilities in the sense of modeling of such sets, typically being fairly complicated. One step in direction of improving the concept of IFS consists in introducing AIFS, a variant of IFS that permits affine invariance property which is vital from the point of modeling. The theory is supported by comprehensive examples.

Keywords: fractals, IFS, AIFS, CAGD properties

1 Introduction

Let $S = \{R^m; w_1, w_2, \dots, w_n\}$ be a hyperbolic iterated function system (IFS) defined on the Euclidean metric space $(\mathbb{R}^m, d_E)$, where

$$w_i(\mathbf{x}) = \mathbf{A}_i\mathbf{x} + \mathbf{b}_i, \quad \mathbf{x} \in \mathbb{R}^m, \quad i = 1, 2, ..., n,$$

$\mathbf{A}_i$ is an $m \times m$ real matrix, $\mathbf{b}_i$ is an m-dimensional real vector and $s_i < 1$, $i = 1, 2, \dots, n$ are the corresponding Lipschitz factors of w_i's. Let $\mathscr{H}(\mathbb{R}^m)$ be the space of nonempty compact subsets of $\mathbb{R}^m$. Let h stand for Hausdorff metric, induced by d_E, i.e.

$$h(A,\, B) = \max\left\{\max_{a\in A}\,\min_{b\in B}\, d_E(a,\, b),\ \max_{b\in B}\,\min_{a\in A}\, d_E(b,\, a)\right\},\ \forall A,\, B \in \mathscr{H}(\mathbb{R}^m),$$

then $(\mathscr{H}(\mathbb{R}^m),\, h)$ is a complete metric space ([1]). Associated with the IFS S is so called *Hutchinson operator* W_S defined on this space by

$$W_S(B) = \bigcup_{i=1}^{n} w_i(B)\,,\ \forall B \in \mathscr{H}(\mathbb{R}^m)\,.$$

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Note that W_S is a contractive mapping on the space $(\mathscr{H}(\mathbb{R}^m), h)$ with contractivity factor $s = \max_i\{ s_i \}$, therefore (by the contraction mapping theorem) W_S has a unique fixed point A, called the *attractor* of the IFS S .

Here are some technical notations. A (non-degenerate) *m-dimensional simplex* $\hat{\mathbf{P}}_m$ (or just *m-simplex*) is the convex hull of a set $\mathbf{P}_m$, of m+1 affinely independent points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{m+1}$ in the Euclidean space of dimension $\geq m$, $\hat{\mathbf{P}}_m = \operatorname{conv}\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{m+1}\}$. Analogously, the *standard* m-simplex is $\hat{\mathbf{T}}_m = \operatorname{conv} \mathbf{T}_m = \operatorname{conv}\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{m+1}\} \subset \mathbb{R}^{m+1}$ where $\mathbf{e}_i$, $i = 1, 2, \ldots, m+1$, are the unit vectors of (m+1)-dimensional orthogonal frame (have all zero-coordinates except 1 at the i-th place). The affine space defined by the standard m-simplex will be denoted by $\mathbb{V}^m$, i.e. $\mathbb{V}^m = \operatorname{aff}\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{m+1}\} \subset \mathbb{R}^{m+1}$.

Let $\mathbf{S} = [s_{ij}]_{i,j=1}^{m+1}$ be an (m+1)×(m+1) row-stochastic real matrix (the matrix which rows sum up to 1).

Definition 1.1 We refer to the linear mapping $\mathscr{L} : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$, such that $\mathscr{L}(\mathbf{x}) = \mathbf{S}^T \mathbf{x}$ as the *linear mapping associated with* $\mathbf{S}$.

Definition 1.2 Let $\hat{\mathbf{P}}_m$ be a non-degenerate simplex and let $\{\mathbf{S}_i\}_{i=1}^n$ be a set of real square nonsingular row-stochastic matrices of order m+1. The system $\Omega(\hat{\mathbf{P}}_m) = \{\hat{\mathbf{P}}_m; \mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_n\}$ is called (*hyperbolic*) *Affine invariant IFS* (*AIFS*), provided that the linear mappings associated with $\mathbf{S}_i$ are contractions in $(\mathbb{R}^m, d_E)$ ([2]-[4]). The AIFS defined in the affine space induced by the standard simplex $\{\hat{\mathbf{T}}_m; \mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_n\}$ is called the *standard AIFS*. The corresponding Hutchinson operator is $W(\mathbf{P}) = \bigcup_i \left(\mathbf{S}_i^T \cdot \mathbf{P}\right)$, for any m-simplex $\mathbf{P}$.

The concept of AIFS is closely related to affine geometry of barycentric (areal) coordinates and subdivision phenomena ([2], [3], [4], [5], [6]). To clear the things up, the following mathematical apparatus is of vital importance.

The affine space $\mathbb{V}^m$ can be treated as a vector space with origin $\mathbf{e}_{m+1}$ provided that $\mathbb{V}^m$ is parallel with the vector space $\mathbf{V}^m = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\} \subset \mathbb{R}^{m+1}$, $\mathbf{u}_i = \mathbf{e}_i - \mathbf{e}_{m+1}$, $i = 1, 2, \ldots, m$. So, from now on, the $\mathbb{V}^m$ will be treated as a vector space unless it is otherwise specified.

The set $\mathscr{U}_m = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ of linearly independent vectors that spans the vector space $\mathbb{V}^m$ is neither orthogonal nor normalized, since $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = \begin{cases} 1, & i \neq j, \\ 2, & i = j. \end{cases}$ By means of the Gram-Schmidt procedure the set $\mathscr{U}_m$ is transformed into an orthonormal basis $\mathscr{V}_m = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ of $\mathbb{V}^m$. This new orthonormal basis is represented by the m×(m+1) matrix

$$\mathbf{V}_m = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1,m+1} \\ v_{21} & v_{22} & & v_{2,m+1} \\ \vdots & & \ddots & \\ v_{m,1} & v_{m,2} & & v_{m,m+1} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \\ \vdots \\ \mathbf{v}_m^{\mathrm{T}} \end{bmatrix}. \tag{1}$$

The relation between the barycentric coordinates of a point $\mathbf{r} \in \mathbb{V}^m$ w.r.t. the affine basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{m+1}\}$, $\mathbf{r} = (\rho_1, \rho_2, \ldots, \rho_{m+1})$, $(\rho_{m+1} = 1 - \sum_{i=1}^m \rho_i)$ and the coordinates in the orthonormal basis $\mathscr{V}_m$, $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, of the same point is given by

$$\begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_m \end{bmatrix} = \begin{bmatrix} v_{11} & v_{21} & \cdots & v_{m,1} \\ v_{12} & v_{22} & & v_{m,2} \\ \vdots & & \ddots & \\ v_{1,m} & v_{2,m} & & v_{m,m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \quad \text{i.e.} \quad \bar{\mathbf{r}} = \mathbf{Q}_m \mathbf{x}, \tag{2}$$

where $\mathbf{Q}_m = \overline{\mathbf{V}_m}^{\mathrm{T}}$, $\overline{\mathbf{V}_m}$ is the truncated matrix $\mathbf{V}_m$, i.e., the matrix $\mathbf{V}_m$ given by (1), with the last column dropped and $\bar{\mathbf{r}}$ is a truncated vector $\mathbf{r}$. The inverse transformation of (2) exists and $\mathbf{x} = \mathbf{Q}_m^{-1}\bar{\mathbf{r}}$ ([5]). In [5], the items of the matrices $\mathbf{Q}_m = [q_{ij}]_{i,j=1}^{m}$ and $\mathbf{Q}_m^{-1} = \left[q'_{ij}\right]_{i,j=1}^{m}$ are explicitly calculated. This result leads to the following

Lemma 1.1 *The $\mathscr{V}_m$-coordinates of the vertices $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_{m+1}$ of the standard simplex $\hat{\mathbf{T}}_m$ are given by the matrix $\mathbf{T}_m^{\mathrm{T}} = [\mathbf{t}_1\ \mathbf{t}_2\ \ldots\ \mathbf{t}_{m+1}] = \left[\mathbf{Q}_m^{-1} \mid 0\right]$. More precisely, $\mathbf{T}_m = [t_{ij}]_{i=1,j=1}^{m,\,m+1}$, where*

$$t_{ij} = \begin{cases} \frac{1}{\sqrt{i(i+1)}}, & j < i \le m, \\ \sqrt{\frac{i+1}{i}}, & i = j \le m, \\ 0, & i < j,\ i = m+1. \end{cases} \tag{3}$$

2 Relation between classical IFS and standard AIFS

Let $\mathbf{x} \in \mathbb{V}^m$ be transformed by an affine transform defined on $\mathbb{V}^m$, given by an $m \times m$ matrix $\mathbf{A}$ and translation vector $\mathbf{b}$, so that

$$\mathbf{x}' = \mathbf{A}\,\mathbf{x} + \mathbf{b}. \tag{4}$$

By (2), $\bar{\mathbf{r}}' = \mathbf{Q}_m\mathbf{x}'$, which by (4) gives $\bar{\mathbf{r}}' = \mathbf{Q}_m\mathbf{x}' = \mathbf{Q}_m(\mathbf{A}\,\mathbf{x} + \mathbf{b})$. Since $\mathbf{x} = \mathbf{Q}_m^{-1}\,\bar{\mathbf{r}}$, the penultimate formula is changed into

$$\bar{\mathbf{r}}' = \mathbf{Q}_m(\mathbf{A}\,\mathbf{Q}_m^{-1}\,\bar{\mathbf{r}} + \mathbf{b}) = \mathbf{Q}_m\mathbf{A}\,\mathbf{Q}_m^{-1}\,\bar{\mathbf{r}} + \mathbf{Q}_m\mathbf{b}. \tag{5}$$

On the other hand, it is easy to see that, if $\mathbf{r} = [\rho_1\ \rho_2\ \ldots\ \rho_{m+1}]^{\mathrm{T}}$ is the "full" areal coordinate vector, then $\bar{\mathbf{r}}$ and $\mathbf{r}$ relate to each other as follows:

$$\bar{\mathbf{r}} \ = \ \mathbf{K}_m \cdot \mathbf{r}\,, \tag{6}$$

$$\mathbf{r} \ = \ \mathbf{J}_m \cdot \bar{\mathbf{r}} \ + \ \mathbf{e}_{m+1}, \tag{7}$$

where

$$\mathbf{K}_m \ = \ \left[\ \mathbf{I}_m \ \mid \ \mathbf{0}\ \right] \qquad \text{and} \qquad \mathbf{J}_m \ = \ \begin{bmatrix} \mathbf{I}_m \\ \hline -\mathbf{1} \end{bmatrix}, \tag{8}$$

are block-matrices obtained by the identity matrix by adding a zero-column or (**–1**) - row. Also $\mathbf{e}_{m+1}$ is the ultimate unit vector in $\mathbb{R}^{m+1}$. So, combining (5), (6) and (7) gives

$$\mathbf{r}' = \tilde{\mathbf{A}}\,\mathbf{r} + \tilde{\mathbf{b}}, \tag{9}$$

where

$$\tilde{\mathbf{A}} = \mathbf{J}_m\mathbf{Q}_m\mathbf{A}\,\mathbf{Q}_m^{-1}\mathbf{K}_m \tag{10}$$

is the $(m+1)\times(m+1)$ matrix, and

$$\tilde{\mathbf{b}} \;=\; \mathbf{J}_m\mathbf{Q}_m\mathbf{b} + \mathbf{e}_{m+1} \tag{11}$$

is the $(m+1)$-dimensional vector. Now the following theorem is valid.

Theorem 2.1 *Let the pair* $(\mathbf{A},\mathbf{b})$ *defines the affine mapping* $\mathbf{x} \mapsto \mathbf{A}\,\mathbf{x} + \mathbf{b}$, $\mathbf{x} \in \mathbb{V}^m$. *Then the corresponding linear transformation* $\mathbf{r} \mapsto \mathbf{S}^T\mathbf{r}$, $\mathbf{r} \in \mathbb{V}^m$ *is defined by the matrix*

$$\mathbf{S}^T \;=\; \tilde{\mathbf{A}} + [\underbrace{\tilde{\mathbf{b}}\;\tilde{\mathbf{b}}\;...\;\tilde{\mathbf{b}}}_{m+1}], \tag{12}$$

where $\tilde{\mathbf{A}}$ *and* $\tilde{\mathbf{b}}$ *are given by* (10) *and* (11).

Proof. Introducing $\mathbf{r} = \mathbf{e}_i$, $i = 1,2,...,m+1$ into (9), the standard simplex vertices $\{\mathbf{e}_1,\, \mathbf{e}_2,\, \dots,\, \mathbf{e}_{m+1}\}$ are being transformed into $\{\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\dots,\boldsymbol{\rho}_{m+1}\}$,

$$\boldsymbol{\rho}_i = \tilde{\mathbf{A}}\,\mathbf{e}_i + \tilde{\mathbf{b}},\;\; i = 1,...,m+1,$$

or, in matrix form

$$[\boldsymbol{\rho}_1\,\boldsymbol{\rho}_2\,...\,\boldsymbol{\rho}_{m+1}] = \tilde{\mathbf{A}}\,[\mathbf{e}_1\,\mathbf{e}_2\,...\,\mathbf{e}_{m+1}] + [\underbrace{\tilde{\mathbf{b}}\;\tilde{\mathbf{b}}\;...\;\tilde{\mathbf{b}}}_{m+1}].$$

Since $[\boldsymbol{\rho}_1\,\boldsymbol{\rho}_2\,...\,\boldsymbol{\rho}_{m+1}] = \mathbf{S}^T$ and $[\mathbf{e}_1\,\mathbf{e}_2\,...\,\mathbf{e}_{m+1}] = \mathbf{I}_{m+1}$, the (12) follows. □

Theorem 2.2 *Given the row stochastic matrix* $\mathbf{S}$*, defining the linear mapping* $\mathbf{r} \mapsto \mathbf{S}^T\mathbf{r}$*,* $r \in \mathbb{V}^m$. *The corresponding linear transformation* $\mathbf{x} \mapsto \mathbf{A}\,\mathbf{x} + \mathbf{b}$, $\mathbf{x} \in \mathbb{V}^m$ *is then given by*

$$\mathbf{A} = \mathbf{Q}_m^{-1}\mathbf{K}_m\mathbf{S}^T\mathbf{J}_m\mathbf{Q}_m,\;\; \mathbf{b} = \mathbf{Q}_m^{-1}\mathbf{K}_m\mathbf{S}^T\mathbf{e}_{m+1}. \tag{13}$$

Proof. Let $\mathbf{r}' = \mathbf{S}^T\mathbf{r}$. Combining it with (6) gives $\bar{\mathbf{r}}' = \mathbf{K}_m\mathbf{S}^T\mathbf{r}$. On the other hand, $\mathbf{x}' = \mathbf{Q}_m^{-1}\bar{\mathbf{r}}'$, so that $\mathbf{x}' = \mathbf{Q}_m^{-1}\mathbf{K}_m\mathbf{S}^T\mathbf{r}$. By inserting (7) one gets

$$\mathbf{x}' = \mathbf{Q}_m^{-1}\mathbf{K}_m\mathbf{S}^T\mathbf{J}_m\bar{\mathbf{r}} \;+\; \mathbf{Q}_m^{-1}\mathbf{K}_m\mathbf{S}^T\mathbf{e}_{m+1}$$

and by (2) $\mathbf{x}' = \left(\mathbf{Q}_m^{-1}\mathbf{K}_m\mathbf{S}^T\mathbf{J}_m\mathbf{Q}_m\right)\mathbf{x} \;+\; \mathbf{Q}_m^{-1}\mathbf{K}_m\mathbf{S}^T\mathbf{e}_{m+1}$, which gives (13). □

3 Applications

The Theorems 2.1 and 2.2 are helpful in connecting two spaces, the designing space $\mathbb{R}^m$ and the affine space $\mathbb{V}^m$. Suppose that our initial model consists of the set of simplices $\{\mathbf{P}; \mathbf{P}_1, \ldots, \mathbf{P}_n\}$ from $\mathbb{R}^m$, where $\mathbf{P}_i = \mathscr{A}_i(\mathbf{P})$ is the image of the simplex $\mathbf{P}$ upon the affine mapping $\mathscr{A}_i$, $i = 1, \ldots, n$. This set accomplishes our construction and fully defines our AIFS (See Def. 1.1 and 1.2), providing the set of linear mappings $\mathscr{L}_i : \mathbf{x} \mapsto \mathbf{S}_i^{\mathrm{T}}\mathbf{x}$, where $\mathscr{L}_i : \mathbf{P} \mapsto \mathbf{P}_i$, is known. On the other hand, the set of subdivision matrices $\{\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_n\}$, can be found by using Theorem 2.1, once the transformation $\mathbf{T}_m^i = \mathscr{A}_i^*(\mathbf{T}_m)$ is being determined. Here, $\mathbf{T}_m$ contains standard simplex's vertices, given as in Lemma 1.1, and $\mathbf{T} = \mathscr{A}(\mathbf{P})$.

Let $\mathbf{P}$, $\mathbf{P}_i$ be any pair of non-degenerate m-simplices and T is the standard m-simplex; $\mathscr{A}$ and $\mathscr{A}_i$ be affine operators defined by $\mathbf{T} = \mathscr{A}(\mathbf{P})$, $\mathbf{P}_i = \mathscr{A}_i(\mathbf{P})$; $\mathbf{T}_i = \mathscr{A}(\mathbf{P}_i)$ and $\tilde{\mathscr{A}_i}$ is defined by $\mathbf{T}_i = \tilde{\mathscr{A}_i}(\mathbf{T})$. If the areal coordinates are in use, then the operators $\mathscr{A}$, $\mathscr{A}_i$ and $\tilde{\mathscr{A}_i}$ can be replaced by linear transformations represented by row-stochastic matrices $\mathbf{S}$, $\mathbf{S}_i$ and $\tilde{\mathbf{S}}_i$ respectively. The next theorem gives relations between affine and linear operators.

Theorem 3.1 *The affine mappings* $\mathscr{A}$, $\mathscr{A}_i$ and $\tilde{\mathscr{A}_i}$ *are connected by the relation*

$$\mathscr{A}_i \circ \mathscr{A} \circ \tilde{\mathscr{A}}_i^{-1} \circ \mathscr{A}^{-1} = \mathscr{I}, \tag{14}$$

where $\mathscr{I}$ *is identity mapping, and* $(f \circ g)(\cdot) = g(f(\cdot))$. *In addition,*

$$\mathbf{S}^{-1} \cdot \tilde{\mathbf{S}}_i^{-1} \cdot \mathbf{S} \cdot \mathbf{S}_i = \mathbf{I}_{m+1}. \tag{15}$$

Proof. By definition, $\mathbf{T}_i = \mathscr{A}(\mathbf{P}_i) = \mathscr{A}(\mathscr{A}_i(\mathbf{P}))$. On the other hand, $\mathbf{T}_i = \tilde{\mathscr{A}_i}(\mathbf{T})$ or $\mathbf{T}_i = \mathscr{A}_i(\mathscr{A}(\mathbf{P}))$. So, $\mathscr{A}(\mathscr{A}_i(\mathbf{P})) = \mathscr{A}_i(\mathscr{A}(\mathbf{P}))$, wherefrom

$$\mathscr{A}(\mathbf{P}) = \tilde{\mathscr{A}}_i^{-1}(\mathscr{A}(\mathscr{A}_i(\mathbf{P}))) \quad \Rightarrow \quad \mathbf{P} = \mathscr{A}^{-1}\left[\tilde{\mathscr{A}}_i^{-1}(\mathscr{A}(\mathscr{A}_i(\mathbf{P})))\right],$$

which, combined with supposition that $\mathbf{P}$ is an arbitrary simplex, yields (14). The relation (15) is direct consequence of (14). □

Example 3.1 *(Variations on Sierpinski triangle).* Consider a non-degenerate 4-simplex $\mathbf{T} = [\mathbf{P}_1\ \mathbf{P}_2\ \mathbf{P}_3\ \mathbf{P}_4\ \mathbf{P}_5]^T$ in $\mathbb{R}^4$ and its images $\mathbf{T}_1$, $\mathbf{T}_2$ and $\mathbf{T}_3$ under three different linear mappings performed by three row-stochastic matrices $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{S}_3$ given in Fig. 1.

The case $\mathbf{T} = [(0, 0)\ (2, 1)\ (0, 3)\ (-2, 1)\ (0, 0)]^T$ is shown in Figure 1a (in all cases the $\mathbb{R}^2$ projections are used). Different configurations of the simplex $\mathbf{T}$ grossly influence the shape of the attractor A. The Fig. 1b is obtained by choosing $\mathbf{T} = [(4, 1)\ (2.75, 2.5)\ (3.5, 4.3)\ (2.25, 3.5)\ (1, 1)]^T$ while the attractor at Fig. 1c is generated for $\mathbf{T} = [(6, 1)\ (6, 6)\ (3.5, 6)\ (1, 6)\ (3, 1)]^T$. But, in all the cases, the basic structure of the fractal set, known as *Sierpinski triangle* (or *gasket*).

$$\mathbf{S}_1=\begin{bmatrix} 1 & 0 & 0 & 0 & 0\\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0\\ \frac{63}{188} & \frac{21}{47} & \frac{7}{47} & \frac{1}{47} & \frac{9}{188}\\ \frac{3}{8} & \frac{3}{16} & \frac{3}{16} & \frac{1}{16} & \frac{3}{16}\\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix},$$

$$\mathbf{S}_2=\begin{bmatrix} \frac{63}{188} & \frac{21}{47} & \frac{7}{47} & \frac{1}{47} & \frac{9}{188}\\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0\\ \frac{9}{188} & \frac{1}{47} & \frac{7}{47} & \frac{21}{47} & \frac{63}{188} \end{bmatrix},$$

$$\mathbf{S}_3=\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2}\\ \frac{3}{16} & \frac{1}{16} & \frac{3}{16} & \frac{3}{16} & \frac{3}{8}\\ \frac{9}{188} & \frac{1}{47} & \frac{7}{47} & \frac{21}{47} & \frac{63}{188}\\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2}\\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

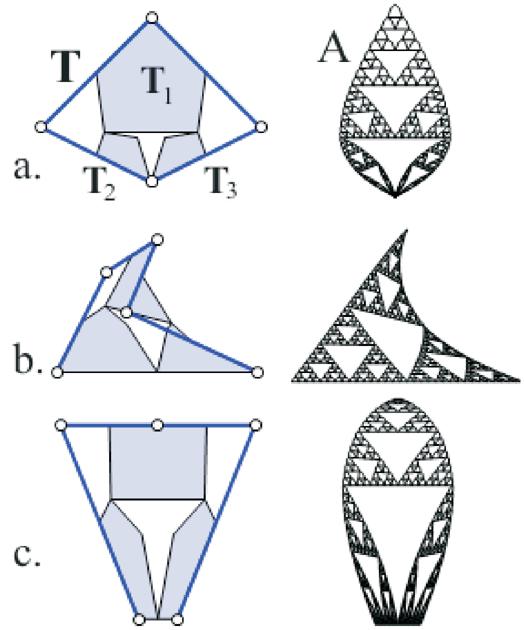

Fig. 1.

Example 3.2 *(Self affine tilling of the domain)*The AIFS method can be used for tessellation of the finite subset of $\mathbb{R}^d$, bounded by a closed hyper-polyhedral surface σ. If σ has m vertices and lies into d-dimensional space ($d \le m-1$), then two possibilities may occur: If $d < m-1$, σ is a projection of m-simplex on $\mathbb{R}^d$; Otherwise, σ is m-simplex itself. Now, suppose that the AIFS Ω=$\{\hat{\mathbf{T}};\mathbf{S}_1,\mathbf{S}_2,\ldots,\mathbf{S}_n\}$ is given, and that $\mathbf{T}_i = \mathbf{S}_i^T\cdot\mathbf{T}$ (all i). Let the AIFS Ω is "just touching", which means that $\bigcup_i \hat{\mathbf{T}}_i=\hat{\mathbf{T}}$, and $\hat{\mathbf{T}}_i\bigcap\hat{\mathbf{T}}_j$ (all i, j) may contain only common border points. Then, the k-th iteration of Hutchinson operator W (see Def. 1.3) applied on $\hat{\mathbf{T}}$, $W^{\circ k}(\hat{\mathbf{T}})$ gives self affine tilling of the domain of $\hat{T}$. Figure 2 illustrates this process in three different settings of tilling the rectangle $\{\mathbf{P}_1,\mathbf{P}_2,\mathbf{P}_3,\mathbf{P}_4\}$ in three parts (first two rows of pictures in Fig. 2) and four parts (the last row). The difference between the first and second partition defined by subdivision matrices $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{S}_3$, is in matrices $\mathbf{S}_1$ and $\mathbf{S}_3$. Namely, for the first setting,

$$\mathbf{S}_1=[\mathbf{P}_4^{\mathrm{T}}\,\mathbf{P}_1^{\mathrm{T}}\,\mathbf{B}^{\mathrm{T}}\,\mathbf{M}^{\mathrm{T}}]^{\mathrm{T}},\ \mathbf{S}_2=[\mathbf{P}_2^{\mathrm{T}}\,\mathbf{B}^{\mathrm{T}}\,\mathbf{M}^{\mathrm{T}}\,\mathbf{P}_3^{\mathrm{T}}]^{\mathrm{T}},\ \mathbf{S}_3=[\mathbf{P}_4^{\mathrm{T}}\,\mathbf{P}_3^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\,\mathbf{P}_4^{\mathrm{T}}]^{\mathrm{T}}$$

while for the second,

$$\mathbf{S}_1=[\mathbf{P}_1^{\mathrm{T}}\,\mathbf{B}^{\mathrm{T}}\,\mathbf{M}^{\mathrm{T}}\,\mathbf{P}_4^{\mathrm{T}}]^{\mathrm{T}},\ \mathbf{S}_2=[\mathbf{P}_2^{\mathrm{T}}\,\mathbf{B}^{\mathrm{T}}\,\mathbf{M}^{\mathrm{T}}\,\mathbf{P}_3^{\mathrm{T}}]^{\mathrm{T}},\ \mathbf{S}_3=[\mathbf{M}^{\mathrm{T}}\,\mathbf{P}_3^{\mathrm{T}}\,\mathbf{P}_4^{\mathrm{T}}\,\mathbf{M}^{\mathrm{T}}]^{\mathrm{T}}.$$

The last row is defined by the setting

$$\begin{aligned}\mathbf{S}_1&=[\mathbf{P}_1^{\mathrm{T}}\,\mathbf{M}^{\mathrm{T}}\,\mathbf{P}_4^{\mathrm{T}}\,\mathbf{P}_1^{\mathrm{T}}]^{\mathrm{T}},\ \ \mathbf{S}_2=[\mathbf{P}_2^{\mathrm{T}}\,\mathbf{M}^{\mathrm{T}}\,\mathbf{P}_1^{\mathrm{T}}\,\mathbf{P}_2^{\mathrm{T}}]^{\mathrm{T}},\\ \mathbf{S}_3&=[\mathbf{P}_3^{\mathrm{T}}\,\mathbf{M}^{\mathrm{T}}\,\mathbf{P}_2^{\mathrm{T}}\,\mathbf{P}_3^{\mathrm{T}}]^{\mathrm{T}},\ \ \mathbf{S}_4=[\mathbf{P}_4^{\mathrm{T}}\,\mathbf{M}^{\mathrm{T}}\,\mathbf{P}_3^{\mathrm{T}}\,\mathbf{P}_4^{\mathrm{T}}]^{\mathrm{T}}.\end{aligned}$$

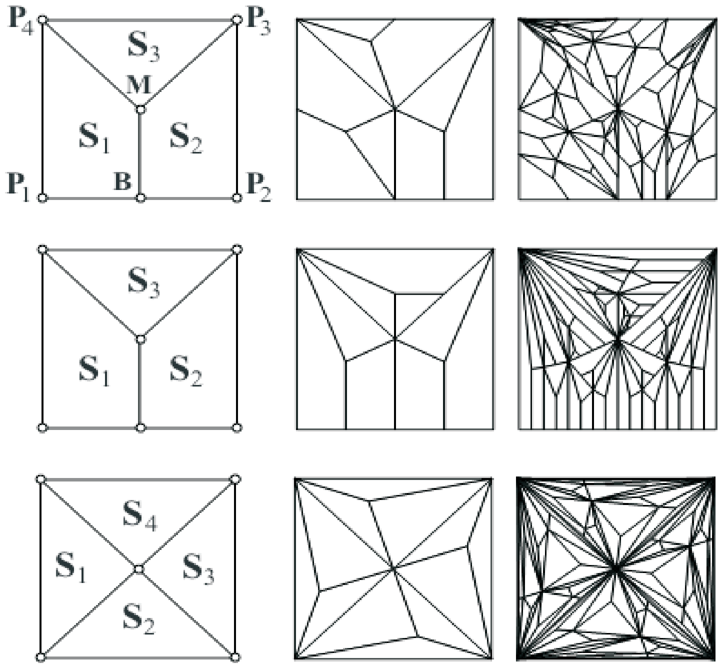

Fig. 2.

The asymmetric subdivision of triangular areas in this case is caused by having two identical rows in each matrix as a consequence of doubling the vertices.

4 Conclusion.

The main result of this note is given in Theorems 2.1 and 2.2, where two new formulas are given for transforming affine into linear operators to connect IFS and AIFS, iterated systems of contractive mappings that are used for defining and constructing fractal sets. A by-result is Theorem 3.1, that gives a rule for both linear and affine mappings that transform a "design" space into referential one and v.v. Two examples are elaborated. The first one is showing the flexibility of the AIFS concept. The second indicates possible applications that are not typically fractal in nature.

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