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On spectral properties of weighted shift operators generated by linear mappings

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Abstract: Weighted shift operators B in the space $L_2(\mathbb{C}^m)$ generated by a linear mapping $A : \mathbb{C}^m \to \mathbb{C}^m$ are considered. A description of properties of $B - \lambda I$ for λ belonging to spectrum $\Sigma(B)$ is given. In particular, a necessary and sufficient condition for $B - \lambda I$ to be one-sided invertible is obtained.

Keywords: weighted shift operators, spectrum, one-sided invertibility, invariant measure, decomposition of oriented graph.

1 Introduction

Let A be a nonsingular linear map in $\mathbb{C}^m$. In the space $L_2(\mathbb{C}^m) = L_2(\mathbb{R}^{2m})$ let us consider weighted shift operators induced by this map, i.e., the operators determined by the expression

$$Bu(x) = a_0(x)u(Ax), \tag{1}$$

where a_0 is a given continuous bounded function.

In more general situation a bounded linear operator B in a Banach space $F(X)$ of functions on a set X is called *weighted shift operator* (WSO) if it can by represented in the form

$$Bu(x) = a_0(x)u(\alpha(x)), \quad x \in X, \tag{2}$$

where $\alpha : X \to X$ is a given map and $a_0(x)$ is a given function on X.

The operators of the form (2), as well as the operator algebras generated by them and related functional equations in different function spaces were studied by a number of authors and have various applications to the theory of dynamical systems, integro-functional, functional-differential, functional and difference equations, nonlocal boundary value problems and in other areas.

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The properties of WSO depend first of all on the dynamics of the map α, i.e. on the behavior of trajectories of points under the action of iterations of this map. Recall that *the trajectory of the point* x_0 is the sequence

$$x_j = \alpha_j(x_0), \quad j \in \mathbb{Z}, \quad \alpha_j(x) = \alpha(\alpha_{j-1}(x)).$$

One of the main problems is to explain the relationship between the dynamics of α and spectral properties of corresponding WSO.

A description of the spectra $\Sigma(B)$ give us the invertibility conditions for the operator $B - \lambda I$. Apart of the invertibility conditions, some subtle properties of the operator $B - \lambda I$ are of a considerable interest such as the closedness of the range, one-sided invertibility, Fredholm and semi-Fredholm property.

In general situation the relationship between the dynamics of α and the subtle spectral properties of WSO is not investigated yet. Earlier such properties were studied only for some special classes of maps α with simple dynamics [1-5]. In the present paper we give a description of the subtle spectral properties for operators (1).

2 Spectrum of weighted shift operator

The problem of describing the spectrum of a weighted shift operator (2) in the case of invertible mapping α in classical spaces is fundamentally solved in sufficient generality.

For the spaces $F(X) = L_2(X, \mu)$ the description of the spectrum look as follows. Let the map α *preserves the class of the measure* μ (i.e., if for a measurable set E equality $\mu(\alpha^{-1}(E)) = 0$ holds if and only if $\mu(E) = 0$). Then there exists a normalizing function ϱ, such that an auxiliary operator of the form $T_\alpha u(x) = \varrho(x)u(\alpha(x))$ is unitary. If X is $\mathbb{C}^m = \mathbb{R}^{2m}$ and α is a diffeomorphism, then $\varrho(x) = |J_\alpha(x)|^{1/2}$, where J_α is the Jacobian of α.

It is more convenient represent the operator in the form of $B = aT_\alpha$, since the properties of the operator B are more simply expressed using the *reduced coefficient*

$$a(x) = \varrho(x)^{-1}a_0(x).$$

For a given map $\alpha : X \to X$ a topological space X is called α-*connected* if cannot be decomposed into two nonempty closed subsets invariant with respect to α. Every connected space is α-connected, but α-connected space may be disconnected.

Theorem 2.1 *Let X be a compact space, μ – a measure on X, whose support coincides with the whole space, $\alpha : X \to X$ – invertible continuous map, preserving the class of measure μ, $a \in C(X)$ and $B = aT_\alpha$. In the space $L_2(X, \mu)$ for the spectral radius $R(B)$ holds*

$$R(B) = \max_{\nu \in M_\alpha(X)} S_\nu(a), \tag{3}$$

where a is the reduced coefficient, $M_\alpha(X)$ is the set of probability measures on X, invariant and ergodic with respect to the map α and

$$S_\nu(a) = \exp[\int_X \ln |a(x)| d\nu] \tag{4}$$

is geometric mean of a with respect to measure ν.

If $a(x) \neq 0$, the set of nonperiodic points of the map α is everywhere dense in X and the space X is α-irreducible, then the spectrum $\Sigma(B)$ coincides with the ring

$$K = \{\lambda \in \mathbb{C} : r(B) \leq |\lambda| \leq R(B)\}, \tag{5}$$

where

$$r(B) = \min_{\nu \in M_\alpha(X)} S_\nu(a).$$

As regards the history and development of these investigations see [6, 7].

3 Compactification of $\mathbb{C}^m$

We cannot directly apply Theorem 2.1 to operator (1), since the space $\mathbb{C}^m$ is not compact, but the study can be reduced to the case of a compact space by using the compactification of the space $\mathbb{C}^m$.

We will consider the compactification of the space $\mathbb{C}^m$ by the sphere at infinity. This compactification arises most clearly by the following construction. The map $x \to \frac{1}{1+\|x\|}x$ is a homeomorphism between the space $\mathbb{C}^m$ and the open unit ball in $\mathbb{C}^m$. The closed unit ball is a compact set in which the open unit ball (homeomorphic to the space $\mathbb{C}^m$) is a dense subset.

Formally, this compactification is constructed as follows. On the set

$$X = \widetilde{\mathbb{C}}^m = \mathbb{C}^m \coprod S^{2m-1}_\infty$$

we introduced a topology. The neighborhood basis of a point $x \in \mathbb{C}^m$ consists of the balls with the center at that point, the neighborhood basis of a point $\xi_0 \in S^{m-1}_\infty$ consists of sets

$$W(\xi_0; R, \delta) = \{x \in \mathbb{C}^m : \|x\| > R,\ \|\frac{1}{\|x\|}x - \xi_0\| < \delta\} \bigcup \{\xi \in S^{m-1}_\infty : \|\xi - \xi_0\| < \delta\},$$

where $R > 0, \delta > 0$. It is easy to check that this topological space is homeomorphic to the closed unit ball and it is a compactification of the space $\mathbb{C}^m$, obtained by adjoining the sphere at infinity S^{2m-1}_∞.

For the described compactification X the algebra $C(X)$ is isomorphic to a functional algebra $C_\infty(\mathbb{C}^m)$ of continuous function on $\mathbb{C}^m$ such that for any $\xi \in S^{2m-1}$ there exists the limit

$$\lim_{t \to +\infty} a(t\xi)$$

and the function

$$\widehat{a}(x) = \begin{cases} a(x), & x \in \mathbb{C}^m, \\ \lim_{t\to+\infty} a(tx), & x \in S_\infty^{m-1}. \end{cases} \tag{6}$$

is continuous on X.

Lemma 3.1 *The algebra $C_\infty(\mathbb{C}^m)$ is invariant with respect to a nonsingular linear map A and there exists a continuous prolongation α of A on X, acting by the formula*

$$\alpha(x) = \begin{cases} Ax, & x \in \mathbb{C}^m, \\ \frac{1}{\|Ax\|}Ax, & x \in S_\infty^{m-1}. \end{cases} \tag{7}$$

In the case of a linear map $\alpha(x) = Ax$ we have $J_\alpha = \det A$, normalizing function $\varrho(x) = |\det A|^{1/2}$ does not depend on x, the operator T_α acts by the formula

$$T_\alpha u(x) = |\det A|^{1/2} u(Ax)$$

and the reduced coefficient is

$$a(x) = \frac{1}{\sqrt{|\det A|}}\, a_0(x). \tag{8}$$

Below, we assume that the operator (1) is written in the form $B = aT_\alpha$ and $a \in C(X)$. Now theorem 2.1 can by applied to operator B.

4 Description of $M(X, \alpha)$

In order to apply theorem 2.1 to operator B, we give a description of the set $M(X, \alpha)$ obtained in [8].

Denote by q the number of different moduli of eigenvalues of A and numerate these moduli in the increasing order

$$0 \le r_1 < r_2 < \ldots < r_q.$$

For given k let $L(k)$ be subspace in $\mathbb{C}^m$, generated by all eigenvectors corresponding to eigenvalues with absolute value r_k. The dimension of the subspace $L(k)$ denote $d(k)$. The intersection $S_k = L(k) \cap S_\infty^{2m-1}$ is a sphere of the dimension $2d(k) - 1$, embedded in the sphere S_∞^{2m-1}, where for different k, the spheres S_k are disjoint. In this way, on the sphere S_∞^{2m-1} is chosen a finite collection of subsets, which are spheres of smaller dimension, invariant with respect to the action of the map α.

If $r_{k_0} = 1$ we denote by $\widetilde{L}_{k_0}$ the compactification of the subspace $L(k_0)$ by infinite sphere S_{k_0}.

Lemma 4.1 *If $r_k \neq 1$ for all k then*

$$Mes(X,\alpha) = \coprod_k Mes(S_k,\alpha) \coprod \{\delta_0\}$$

where δ_0 is the measure concentrated at the point 0.

If $r_{k_0} = 1$ then

$$Mes(X,\alpha) = \coprod_{k \neq k_0} Mes(S_k,\alpha) \coprod Mes(\widetilde{L}_{k_0},\alpha).$$

On the set S_k and on $\widetilde{L}_{k_0}$ the mapping α acts as some unitary operator $U(k)$ and the problem is reduced to the description of the measures ν on $\mathbb{C}^d$, invariant and ergodic with respect to an unitary operator.

Let us consider a unitary operator U in a finite-dimensional space. Without a loss in generality, we may assume that the space under consideration is $\mathbb{C}^d$ and that the operator is given by a diagonal matrix:

$$U = diag\{\omega_1, \omega_2, \ldots, \omega_d\}, \quad \text{where} \quad \omega_j = e^{i2\pi h_j}, \ h_j \in \mathbb{R}. \tag{9}$$

Consider the map

$$\varphi(x) = (|x_1|, |x_2|, \ldots, |x_d|). \tag{10}$$

This map acts from $\mathbb{C}^d$ to $\mathbb{R}^d$ and the image is the closed cone of positive vectors $\mathbb{R}^d_+$. Since obviously $\varphi(Ux) = \varphi(x)$, the preimage of every point from $\mathbb{R}^{d+}$ is a nonempty close invariant set.

The preimage of a point $\xi \in \mathbb{R}^d_+$ has the form

$$\varphi^{-1}(\xi) = \{x \in \mathbb{C}^d : |x_k| = \xi_k\}.$$

It is obvious that if $\xi_k = 0$, then $x_k = 0$, and if $\xi_k \neq 0$, then $x_k = z_k \xi_k$, where $|z_k| = 1$.

In this way, the points from $\varphi^{-1}(\xi)$ may be naturally parameterized by the collection of numbers z_k with indices k, for which $\xi_k \neq 0$, and satisfying the condition $|z_k| = 1$. This means that the set $\varphi^{-1}(\xi)$ is homeomorphic to the product of finitely many circles, i.e., it is a torus, which we denote by $\mathbf{T}_\xi$. The dimension of that torus is equal to the number of nonzero coordinates of the vector ξ, and we denote that dimension by $d(\xi)$.

In particular, for interior points ξ of the set $\mathbb{R}^d_+$ all coordinates are nonzero and the dimension of the torus $\mathbf{T}_\xi$ is d; for boundary points corresponding tori have dimension from 1 to $d-1$. To the point 0 corresponds the degenerate torus consisting of only one point.

Any torus $\mathbf{T}_\xi$ is a group with respect to the operation of coordinate-wise multiplication and the identity element is $(1, 1, \ldots, 1)$. Every torus T_ξ is invariant with respect to the action of the operator U and the action of the map U on T_ξ is given as coordinate-wise multiplication by the vector $\omega(\xi)$, formed by the numbers ω_j with index j, for which $\xi_j \neq 0$. Such a map of torus into itself is called *standard shift of torus.*

Let us denote by $H_{\omega(\xi)}$ the closure in $\mathbf{T}_\xi$ of the set $\{\omega(\xi)^n, n \in \mathbb{Z}\}$. Then $H_{\omega(\xi)}$ is a closed subgroup of the torus $\mathbf{T}_\xi$ and on $H_{\omega(\xi)}$ there exists a uniquely determined normalized Haar measure ν_ξ.

Lemma 4.2 *Under the action of the unitary operator U, given by (9), the space $\mathbb{C}^d$ stratifies by the map*

$$\varphi(x) = (|x_1|, |x_2|, \dots, |x_d|)$$

into invariant tori $\boldsymbol{T}_\xi$, parameterized by the points ξ from the positive cone $\mathbb{R}^d_+$.

For arbitrary measure ν on $\mathbb{C}^d$ invariant and ergodic with respect to U there exist $\xi \in \mathbb{R}^d_+$ and $x \in \boldsymbol{T}_\xi$ such that ν is supported on equivalence class $[x]$ from the quotient group $\boldsymbol{T}_\xi / H_{\omega(\xi)}$ and the geometric mean (4) can be written in the form

$$S_\nu(a) = exp[\int_{H(\omega(\xi))} \ln |a(zx)| d\nu_\xi(z)].$$

These lemmas give us a description of the set $M(X, \alpha)$.

5 Subtle spectral properties

If $r_k \neq 1$ let us denote

$$R_k^+(a) = \max_{\nu \in Mes(S_k, \alpha)} S_\nu(a),$$

$$R_k^-(a) = \min_{\nu \in Mes(S_k, \alpha)} S_\nu(a).$$

and

$$R_0^\pm(a) = |a(0)|.$$

If $r_{k_0} = 1$ we denote

$$R_{k_0}^+(a) = \max_{\nu \in Mes(\widetilde{L}_{k_0}, \alpha)} S_\nu(a),$$

$$R_{k_0}^-(a) = \min_{\nu \in Mes(\widetilde{L}_{k_0}, \alpha)} S_\nu(a).$$

The sets

$$\Sigma_k = \{\lambda : R_k^-(a) \le |\lambda| \le R_k^+(a)\}$$

belong to the spectrum $\Sigma(B)$ and are some closed rings (may be degenerated to circles). The set

$$\Sigma(B) \setminus \bigcup \Sigma_k$$

is an union of some open subrings. In this way we construct decomposition of $\Sigma(B)$ into a family of subrings. The study shows that the properties of the operator $B-\lambda I$ are the same for λ from a fixed subring, and may be different for λ from different subrings. Therefore, the problem consists of the description of properties of the operator $B - \lambda I$ for different subrings.

The answer depends on dynamics of α and the majority/minority relationships between the numbers $R_k^{\pm}(a)$ and $|\lambda|$. A characteristic of the map α which was useful in obtaining solutions of the problem in question turned out to be an *oriented graph* $G_\alpha(X)$ with the vertices W_k, describing the dynamics of the map.

If $r_k \neq 1$ for all k we put $W_k = S_k$, $k = 0, 1, \ldots, q$

If $r_{k_0} = 1$ we put $W_k = S_k$ for $k = 1, \ldots, k_0-1, k_0+1, \ldots, q$ and put $W_{k_0} = \widetilde{L}_{k_0}$.

An *oriented edge* $W_k \to W_j$ is included in the graph if and only if there exists a point $x \in X$ such that its trajectory tends to W_j as $n \to +\infty$ and tends to W_k as $n \to -\infty$.

Lemma 5.1 *Let $r_k \neq 1$ for all k and k' is such a number that $r_{k'} < 1 < r_{k'+1}$. Then the graph $G_\alpha(X)$ has the form*

$$W_1 \to W_2 \to \ldots \to W_{k'} \to W_0 \to W_{k'+1} \to \ldots \to W_q.$$

If there exists k_o such that $r_{k_0} = 1$, then the graph $G_\alpha(X)$ has the form

$$W_1 \to W_2 \to \ldots \to W_{k_0} \to W_{k_0+1} \to \ldots \to W_q.$$

Using the coefficient a and number λ one forms two subsets of the set of vertices of the graph

$$G^+(a,\lambda) = \{W_k : R_k^-(a) > |\lambda|\}, \tag{11}$$
$$G^-(a,\lambda) = \{W_k : R_k^+(a) < |\lambda|\}.$$

It is clear that

$$G^+(a,\lambda) \bigcap G^-(a,\lambda) = \emptyset.$$

We say that subsets $G^+(a,\lambda)$ and $G^-(a,\lambda)$ give a*decomposition of the graph*, if the condition

$$G_\alpha(X) = G^+(a,\lambda) \bigcup G^-(a,\lambda)$$

holds. This condition is equivalent to the condition

$$|\lambda| \neq S_\nu(a) \quad \text{for all} \quad \nu \in Mes(X,\alpha).$$

The graph decomposition will be called *oriented to the right* (*oriented to the left*) if any edge connecting the point $W_k \in G^-(a,\lambda)$ to the point $W_j \in G^+(a,\lambda)$, is oriented from W_k to W_j (is oriented from F_j to F_k).

The main result of the paper is the following theorem.

Theorem 5.2 *Let $A : \mathbb{C}^m \to \mathbb{C}^m$ be nongenerated linear map, B be weighted shift operator in the space $L_2(\mathbb{C}^m)$ of the form (1), coefficient $a \in C(X)$, where X is the compactification of the space $\mathbb{C}^m$ by the sphere at infinity, and let $a(x) \neq 0$ for all $x \in X$.*

The operator $B - \lambda I$ is invertible from the right (left) if and only if the subsets $G^+(a, \lambda)$ and $G^-(a, \lambda)$ form a decomposition of the graph G_α which is oriented to the right (to the left).

The image of the operator $B - \lambda I$ is closed if and only if this operator is one-sided invertible.

An analogous Theorem was obtained in [5] for weighted shift operator, generated by so-called Morse – Smale type mapping α. The proof of Theorem 5.2 is similar the one from [5], but we have a mapping with more complicated dynamic and all calculations are more complicated.

References

[1] Yu.I. Karlovich , R. Mardiev , *One -sided invertibility of functional operator with non-Carleman shift in Hölder spaces,* Sovet Math. (Iz.VUZ), 31 ,3(1987), 106-110.

[2] G. Belitskii, Yu.Lyubich, *On the normal solvability of cohomological equations on locally compact topological spaces // Nonlinear analysis and related problems*, Trudy Inst.Mat. Minsk, 2(1999), 44-51.

[3] A.Yu.Karlovich, Yu.I.Karlovich, *One sided invertibility of binomial functional operators with a shift in rearrangement-invariant spaces.* Integral Equations Operator Theory, 42(2002), 201-228.

[4] A.B. Antonevich, *Coherent Local Hyperbolicity of a Linear Extension and Essential Spectra of a Weighted Shift Operator on a Closed Interval,* Functional Analysis and Its Appl., 39, 1 (2005), 9-20, 2005.

[5] A. Antonevich, Yu. Makowska, *On spectral properties of weighted shift operators generated by mappings with saddle points* , Complex analysis and Operator theory, 2 (2008), 215-240.

[6] A.Antonevich *Linear Functional Equations. Operator Approach*, Operator Theory: Advances and Applications, Vol.83, Birkhäuser, Basel, 1996.

[7] A. Antonevich, A. Lebedev, *Functional Differential Equations: I. C^*-theory*, Longman, Harlow, 1994.

[8] A. Antonevich, A.Buraczewski, *Dynamics of linear mapping and invariant measures on sphere*, Demonstratio Math., 29,4 (1996), 817-824.