SCIENTIFIC PUBLICATIONS OF THE STATE UNIVERSITY OF NOVI PAZAR
SER. A: APPL. MATH. INFORM. AND MECH. vol. 4, 1 (2012), 53-60

AIFS and Newton's interpolating polynomials

Lj. M. Kocić, S. Gegovska – Zajkova, V. Andova

Abstract: A multi-segment subdivision scheme for an arbitrary univariate real polynomial on a finite interval is established. The method uses Newton's interpolation in combination with the AIFS (Affine invariant Iterated Function Systems) to construct a fractal-type algorithm that products polynomial geometry.

Keywords: Newton's interpolation polynomial, IFS, AIFS

1 Introduction

The Iterated Function Systems (IFS) and its affine invariant counterpart AIFS are very powerful tool for fractal sets modeling. In the case when the collection of objects to be modeled, besides fractals contains smooth objects, such as polynomials, we have to introduce new algorithms capable to create both fractal and smooth forms. The aim of this paper is to develop such algorithms for Newton's interpolating polynomials.

Let $\{\omega_i, i = 1, \ldots, n\}$, $n > 1$ be a set of contractive affine mappings defined on the complete Euclidian metric space $(\mathbb{R}^m, d_E)$

$$\omega_i(\mathbf{x}) = \mathbf{A}_i\mathbf{x} + \mathbf{b}_i, \ \ \mathbf{x} \in \mathbb{R}^m, \ i = 1, \ldots, n, \tag{1}$$

where $\mathbf{A}_i$ is an real matrix and $\mathbf{b}_i$ is an m-dimensional real vector. Supposing that the Lipschitz factors $s_i = Lip\{\omega_i\}$, satisfy condition $|s_i| < 1, i = 1, \ldots, n$, the system $\{\mathbb{R}^m; \omega_1, \omega_2, \ldots, \omega_n\}$ is called (hyperbolic) Iterated Function System (IFS). Associated with given IFS, is so called Hutchinson operator $\mathscr{H}(\mathbb{R}^m) \to \mathscr{H}(\mathbb{R}^m)$, defined by

$$W(B) = \bigcup_{i=1}^{n} w_i(B), \quad \forall B \in \mathscr{H}(\mathbb{R}^m). \tag{2}$$

Manuscript received October 4, 2011; revised December 15, 2011; accepted January 17, 2012.

Ljubiša M. Kocić is with the University of Nis, Faculty of Electronic Engineering, Nis, Serbia; Sonja Gegovska-Zajkova and Vesna Andova are with the Faculty of Electrical Engineering and Information Technologies, University Ss Cyril and Methodius, Skopje, Macedonia

It turns to be a contractive mapping on the complete metric space$(\mathscr{H}(\mathbb{R}^m),h)$ with contractivity factor $s=\max\{s_i\}$ and $\mathscr{H}(\mathbb{R}^m)$ is the space of nonempty compact subsets of $\mathbb{R}^m$. By h we denote Hausdorff metric induced by d_E , i.e.

$$h(A,B)=\max\left\{\max_{a\in A}\min_{b\in B}d_E(a,b),\max_{b\in B}\min_{a\in A}d_E(b,a)\right\},\quad \forall A,B\in\mathscr{H}(\mathbb{R}^m).$$

Let $S_{m+1}=[s_{ij}]_{i,j=1}^{m+1}$ be a row-stochastic real matrix (its rows sum up to 1).

Definition 1.1 We refer to the linear mapping $\mathscr{L}:\mathbb{R}^{m+1}\to\mathbb{R}^{m+1}$, such that $\mathscr{L}(\mathbf{x})=\mathbf{S}^T\mathbf{x}$ as the *linear mapping associated with* $\mathbf{S}$.

The corresponding Hutchinson operator is

$$W(B)=\bigcup_{i=1}^{n}\mathscr{L}_i(B),\quad \forall B\in\mathscr{H}(\mathbb{R}^{m+1}).\tag{3}$$

According to the contraction mapping theorem, both Hutchinson operators (2) and (3) have the unique fixed point called the attractor of the IFS/AIFS. In the case of AIFS, the attractor $A\in\mathscr{H}(\mathbb{R}^{m+1})$ satisfies $A=W(A)$.

Definition 1.2 A (non-degenerate) *m-dimensional simplex* $\hat{\mathbf{P}}_m$ (m-simplex) is a convex hull $\hat{\mathbf{P}}_m=\operatorname{conv}\{\mathbf{P}_m\}$ of a set $\mathbf{P}_m$ of $m+1$ affinely independent points (vectors) $\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_{m+1}$ in Euclidean space of dimension $\geq m$ that will be denoted in matrix form

$$\mathbf{P}_m=\left[\mathbf{p}_1^{\mathrm{T}}\ \mathbf{p}_2^{\mathrm{T}}\ \cdots\ \mathbf{p}_{m+1}^{\mathrm{T}}\right]^{\mathrm{T}}.$$

Definition 1.3 Let $\hat{\mathbf{P}}_m$ be a non-degenerate simplex and let $\{\mathbf{S}_i\}_{i=1}^n$ be a set of real square nonsingular row-stochastic matrices of order m+1. The system $\Omega(\hat{\mathbf{P}}_m)=\{\hat{\mathbf{P}}_m;\mathbf{S}_1,\mathbf{S}_2,\ldots,\mathbf{S}_n\}$ is called (*hyperbolic*) *Affine invariant IFS* (*AIFS*), provided that the linear mappings associated with $\mathbf{S}_i$ are contractions in ($\mathbb{R}^m$, d_E) ([5]-[7]).

Theorem 1.1 *One eigenvalue of the matrix* $\mathbf{S}_i$ *is* 1*, other m eigenvalues coincide with eigenvalues of* $\mathbf{A}_i$*, the matrix that makes the linear part of the affine mapping* ω_i *given by* (1).

2 Newtonian subdivision

The notion of subdivision is usually attributed to m-dimensional $(m\geq 1)$ continuous parametric mapping $t\mapsto P_n(t)$, $t\in[a,b],(a<b)$, so that $P_n(t)\in\mathbb{R}^m$. Thus, to study the basic properties, it is enough to consider one-dimensional case $(m=1)$.

Let

$$P_n(t) = \sum_{k=0}^{n} A_k B_k^n(t), \quad t \in [a,b], \tag{4}$$

be a function basis, where A_k are real coefficients and $\mathscr{B}_n(t) = \{B_0^n(t), \ldots, B_n^N(t)\}$, $t \in [a,b]$. Both A_k and $\mathscr{B}_n(t)$ may depend on the definition interval. To stress this fact, it is suitable to write $A_k[a,b]$ as well as $\mathscr{B}_n[a,b](t)$.

Definition 2.1 The function P_n, defined by (4) is said to permit linear subdivision if and only if for each nonempty subinterval $[p,q] \subset [a,b]$, there exists a set of coefficients $\{A_k[p,q]\}_{k=0}^n$ such that

$$\sum_{k=0}^{n} A_k[p,q] B_k^n[p,q](t) = \sum_{k=0}^{n} A_k[a,b] B_k^n[a,b](\varphi(t)),$$

for $t \in [a,b]$, where

$$\varphi(t) = \frac{1}{b-a}((q-p)t + bp - aq)$$

maps $[a,b]$ onto $[p,q]$.

According to Goldman and Heath ([4]), linear subdivision is strictly a polynomial phenomenon.

Theorem 2.1 ([4]) *The function* $P_n(t)$, *defined by* (4) *admits linear subdivision if and only if* $\mathscr{B}_n(t)$ *is a polynomial basis.*

The best known subdivision phenomena is connected with Bernstein polynomial basis, but subdivision is also possible for monomial, Lagrange, Newton's or any other polynomial basis ([1], [9]). Here, we will focus on multi-segment subdivision for Newton's interpolation polynomials.

Finding an analytic, preferably polynomial representation of a curve or a surface described by a non-analytic (sometimes even a non-mathematical) way is a very important task in computer-aided modeling and other applications. One of the most popular methods is Newton's polynomial interpolation scheme. It is a numerical tool that assigns an algebraic polynomial to the set of discrete data. Polynomials have a lot of advantages. They are easy to process operationally (division, multiplication, differentiation, integration etc), they are free of poles and stable from the numerical point of view. The scheme itself is easy to implement in the form of computer program, and simple to use.

The Newton's interpolation polynomial needs divided differences of the data instead the data itself. Let us consider a set of data given by the plane points, called nodes, that we will identify by vectors

$$\mathbf{P}_i = (x_i, y_i) = [x_i \; y_i]^{\mathrm{T}}, \quad i = 0, \ldots, n \; (n \geq 2), \tag{5}$$

where $x_i < x_{i+1}$. The vector of divided differences $\nabla\mathbf{y}$ is given by

$$\nabla^0 y_i = y_i, \quad \nabla^k y_i = \frac{\nabla^{(k-1)} y_{i+1} - \nabla^{(k-1)} y_i}{x_{i+k} - x_i},$$

and thus

$$\nabla\mathbf{y} = \left[\nabla^0 y_0 \ \nabla^1 y_0 \ \dots \ \nabla^n y_0\right]^{\mathrm{T}}.$$

Newton's interpolation polynomial is given by

$$\mathbf{N}_n(x) = \sum_{k=0}^{n} \nabla^k y_0 v_k^n(x) = (\nabla\mathbf{y})^{\mathrm{T}} \cdot \mathbf{v}^n(x),$$

where $\mathbf{v}^n(x) = [v_0^n(x) \ v_1^n(x) \ \cdots \ v_n^n(x)]^{\mathrm{T}}$ is the vector of Newton's basis functions

$$v_0^n(x) = 1, \quad v_k^n(x) = \prod_{i=0}^{k-1} (x - x_i), \quad x \in [x_0, x_n]. \tag{6}$$

Put $a = x_0$, $b = x_n$ and choose $c \in (a, b)$, the point that devides $[a, b]$ in the ratio $\lambda = \dfrac{c-a}{b-a}$. Consider two affine mappings: the left one $\varphi_{\mathrm{L}} : [a, b] \to [a, c]$ and right one $\varphi_{\mathrm{R}} : [a, b] \to [c, b]$, given by

$$\varphi_{\mathrm{L}} = \frac{c-a}{b-a} x + \frac{b-c}{b-a} a \quad \text{and} \quad \varphi_{\mathrm{R}} = \frac{b-c}{b-a} x + \frac{c-a}{b-a} b$$

and apply both of them on the set of abscissas of the interpolation data $\mathbf{x} = \{x_0, x_1, \dots, x_n\}$ to get "left" image

$$\mathbf{x}^{\mathrm{L}} = \varphi_{\mathrm{L}}(\mathbf{x}) = \left\{x_0^{\mathrm{L}}, x_1^{\mathrm{L}}, \dots, x_n^{\mathrm{L}}\right\}$$

and the "right" one

$$\mathbf{x}^{\mathrm{R}} = \varphi_{\mathrm{R}}(\mathbf{x}) = \left\{x_0^{\mathrm{R}}, x_1^{\mathrm{R}}, \dots, x_n^{\mathrm{R}}\right\}.$$

The corresponding data ordinates are

$$\mathbf{y} = \{y_0, \dots, y_n\}, \quad \mathbf{y}^{\mathrm{L}} = \left\{y_0^{\mathrm{L}}, y_1^{\mathrm{L}}, \dots, y_n^{\mathrm{L}}\right\} \quad \text{and} \quad \mathbf{y}^{\mathrm{R}} = \left\{y_0^{\mathrm{R}}, y_1^{\mathrm{R}}, \dots, y_n^{\mathrm{R}}\right\},$$

where the "left" ordinates can be expressed by

$$y_i^{\mathrm{L}} = \mathbf{N}_n\left(x_i^{\mathrm{L}}\right) = \mathbf{y}^{\mathrm{T}} \cdot \mathbf{v}^n\left(x_i^{\mathrm{L}}\right), \, i = 0, 1, \dots, n,$$

or in the matrix form

$$\mathbf{y}^{\mathrm{L}} = \begin{bmatrix} v_0^n\left(x_1^{\mathrm{L}}\right) & v_1^n\left(x_1^{\mathrm{L}}\right) & \cdots & v_n^n\left(x_1^{\mathrm{L}}\right) \\ v_0^n\left(x_2^{\mathrm{L}}\right) & v_1^n\left(x_2^{\mathrm{L}}\right) & \cdots & v_n^n\left(x_2^{\mathrm{L}}\right) \\ & & \vdots & \\ v_0^n\left(x_n^{\mathrm{L}}\right) & v_1^n\left(x_n^{\mathrm{L}}\right) & \cdots & v_n^n\left(x_n^{\mathrm{L}}\right) \end{bmatrix} \cdot \begin{bmatrix} \nabla^0 y_0 \\ \nabla^1 y_0 \\ \vdots \\ \nabla^n y_0 \end{bmatrix} = \mathbf{N}_{\mathrm{L}}^{\mathrm{T}} \cdot \nabla\mathbf{y}, \tag{7}$$

where

$$\mathrm{N_L} = \left[v_i^n\left(x_j^{\mathrm{L}}\right)\right]_{\substack{i=0,n\\ j=0,n}}.$$

If the data ordinates are components of the vector $\mathbf{y}$, then

$$\nabla \mathbf{y} = \mathrm{Q}\cdot \mathbf{y}, \tag{8}$$

where

$$\mathrm{Q} = [q_{ij}]_{\substack{i=0,n\\ j=0,n}}, \quad \frac{1}{q_{ij}} = \frac{d}{dx}v_{i+1}^n(x)\bigg|_{x=x_i} \tag{9}$$

is a square $(n+1)$-order matrix. Combination of (7) and (8) yields $\mathbf{y}^{\mathrm{L}} = \mathrm{N_L^T}\cdot\nabla\mathbf{y} = \left(\mathrm{N_L^T}\cdot \mathrm{Q}\right)\cdot \mathbf{y} = \mathrm{S_L}\cdot\mathbf{y}$. Now, we obtain subdivision matrices

$$\mathrm{S_L} = \mathrm{N_L^T}\cdot \mathrm{Q}, \quad \mathrm{S_R} = \mathrm{N_R^T}\cdot \mathrm{Q}, \tag{10}$$

where

$$\mathrm{N_R} = \left[v_i^n\left(x_j^{\mathrm{R}}\right)\right]_{\substack{i=0,n\\ j=0,n}}.$$

Example 2.1 Let the interval $[a, b] = [-2, 2]$ and the set of data is given by

$$\mathbf{x} = [-2 \quad -1 \quad -0.5 \quad 0.5 \quad 1 \quad 2]^{\mathrm{T}} \quad \text{and} \quad \mathbf{y} = [2 \quad 1 \quad -1 \quad 0.7 \quad 0.7 \quad 1]^{\mathrm{T}}.$$

The choice $c = 0.4 \in [-2, 2]$ gives the subdivision factor $\lambda = 0.6$. The corresponding Newton basis is

$$\mathbf{v}^n = \begin{array}{l}[1 \quad (x-x_1) \quad (x-x_1)(x-x_2) \quad (x-x_1)(x-x_2)(x-x_3)\\ (x-x_1)(x-x_2)(x-x_3)(x-x_4) \quad (x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)]^{\mathrm{T}}.\end{array}$$

To evaluate the divided difference vector, it is suitable to fix the matrix Q by using (9) and then apply (8). The direct calculations results in

$$\mathrm{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ -1 & 1 & 0 & 0 & 0 & 0\\ 2/3 & -2 & 4/3 & 0 & 0 & 0\\ -4/15 & 4/3 & -4/3 & 0 & 0 & 0\\ 4/15 & -2/3 & 8/9 & -8/15 & 2/9 & 0\\ -1/45 & 2/9 & -16/45 & 16/45 & -2/9 & 1/45 \end{bmatrix}$$

so that $\nabla\mathbf{y} = \mathrm{Q}\cdot\mathbf{y} = [2 \quad -1 \quad -2 \quad 58/25 \quad -359/225 \quad 146/225]^{\mathrm{T}}$, which reveals the form of the Newton's interpolating polynomial

$$\begin{aligned}\mathbf{N}_n \;=\;& 2-(x+2)-2(x+2)(x+1)+\frac{58}{25}(x+2)(x+1)(x+0.5)\\ &-\frac{359}{225}(x+2)(x+1)(x+0.5)(x-0.5)\\ &+\frac{145}{225}(x+2)(x+2)(x+0.5)(x-0.5)(x-1).\end{aligned} \tag{11}$$

Thus, the subdivision matrices are

$$S_{\mathrm{L}} = N_{\mathrm{L}} \cdot Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.1240 & 1.8605 & -1.3230 & 0.6267 & -0.3101 & 0.0219 \\ 0.0139 & 1.2499 & -0.3333 & 0.1250 & -0.0595 & 0.0040 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0.0098 & -0.2217 & 0.9462 & 0.4055 & -0.1478 & 0.0080 \\ 0.0027 & -0.0461 & 0.1147 & 1.0322 & -0.1075 & 0.0040 \end{bmatrix}$$

and

$$S_{\mathrm{R}} = N_{\mathrm{R}} \cdot Q = \begin{bmatrix} 0.0027 & -0.0461 & 0.1147 & 1.0322 & -0.1075 & 0.0040 \\ -0.0037 & 0.0582 & -0.1290 & 0.5591 & 0.5241 & 0.0012 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0.0219 & -0.3101 & 0.6267 & -1.3230 & 1.8605 & 0.1240 \\ 0.0320 & -0.4435 & 0.8786 & -1.6773 & 1.9219 & 0.2883 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

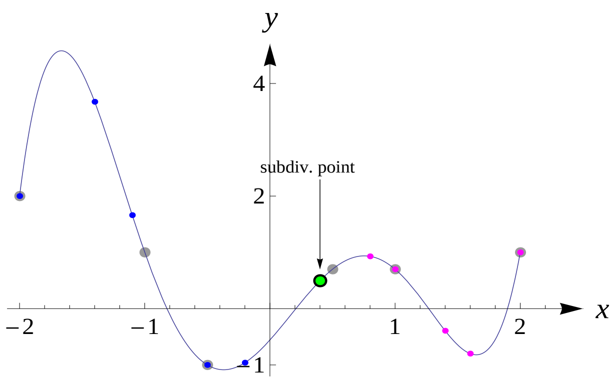

Fig.1. Binary subdivision of Newton's basis.

The following main result is a trivial generalization of the above considerations. Let $(x_i, y_i)_{i=0,n}$, $(x_i < x_{i+1})$, $x_i \in [a, b]$ be the set of interpolation data. Let $I_1, \ldots, I_m$ be a partition of $[a, b]$ and let φ_s be the increasing affine function that maps $[a, b] \to I_s$.

Theorem 2.2 *(Newtonian subdivision) The AIFS* $\{\hat{\mathbf{P}}_{n+1}; \mathbf{N}_1^{\mathrm{T}}\mathbf{Q}, \ldots, \mathbf{N}_m^{\mathrm{T}}\mathbf{Q}\}$, *where* $\mathbf{N}_k = \left[\mathbf{v}_i^n\left(x_i^k\right)\right]_{\substack{i=0,n \\ j=0,n}}$, $k = 1, \ldots, m$, $\mathbf{v}_i^n$ *is given by* (6), $x_i^s = \varphi_s(x_j)$, $j = 0, \ldots, n$ *and* Q *is matrix* (9) *has the graph of the n-the degree polynomial interpolating the data set* $(x_i, y_i)_{i=0,n}$ *as its attractor.*

3 The attractor

The subdivision matrices from the AIFS defined in Theorem 2.2, $S_s = \mathbf{N}_s^{\mathrm{T}} \cdot \mathbf{Q}$, define linear mappings in $\mathbb{R}^{m+1}$, through the Hutchinson operator (2)

$$W'(\mathbf{x}) = \bigcup_{s=1}^{m} S_s(\mathbf{x}).$$

Going back to the Example 2.1, where $m = 2$, the AIFS contains two mappings embodied in matrices S_{L} and S_{R}. Then, the random-type algorithm ([2], [3]) could be applied for graphical rendering of the object of subdivision, which is the graph of polynomial curve. In fact by the random algorithm one can calculates orbit of W', i.e. the set $(W')^{\circ m}(\mathbf{x}_0)$ where $\mathbf{x}_0 \in \mathbb{R}^{m+1}$.

The Newton's basis is just slightly different in evaluation of the subdivision matrices, but the matrices will be the same as in Lagrange case, depending just on the interpolating points and the ratio of subdivision of the interval $[a, b]$. In the Figure 1 the graph of the interpolant (11) is shown. The result of applying random algorithm on the Newton's polynomial using 200, i.e. 1000 iterations is depicted in Figure 2.

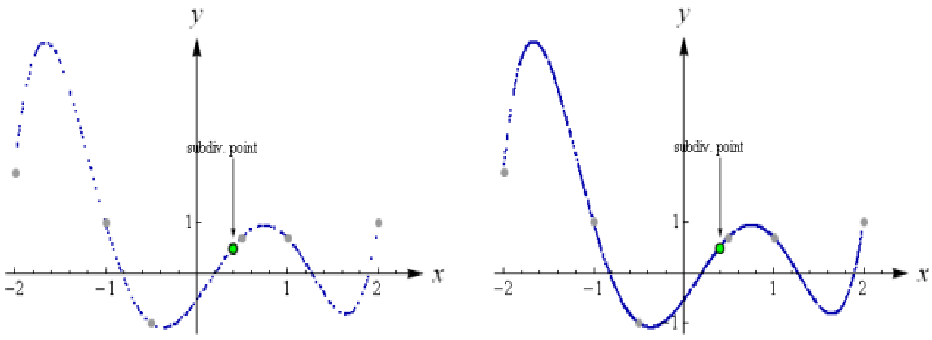

Fig.2. Newton's interpolant as an attractor. Left: 200 points; Right: 1000 points.

4 Conclusion

As soon as fractal sets have been recognized as a very important mathematical tool for modeling fuzzy and complex objects that the Nature is full of, it rises the problem of establishing a connection between this new realm and the world of classic mathematical objects. This paper is an attempt to trace a method for constructing subdivision processes for Newton's polynomial bases. The process is embodied in subdivision matrices that further enable construction of the AIFS systems and apply the fractal-oriented algorithms for rendering the corresponding polynomial objects. Many further questions are opened and left for future research. The main question is to find the AIFS for any given basis, then there is question of best algorithm to be applied, etc.

References

[1] V. ANDOVA: *A method for polynomial attractors construction*. Master Thesis, (in Macedonian), University Ss Cyril and Methodius, Skopje, Macedonia, 2009.

[2] M. F. BARNSLEY: *Fractals Everywhere*. Academic Press, San Diego, 1993.

[3] S. DUBUC, A. ELQORTOBI: *Approximation of fractal sets*. J. Comput. Appl. Math., 29 (1990), 79-89.

[4] R. N. GOLDMAN, D. C. HEATH: *Linear subdivision is strictly a polynomial phenomenon*. Comput. Aided. Geom. Des. 1 (1984), 269-278.

[5] LJ. M. KOCIĆ, A. C. SIMONCELLI: *Shape predictable IFS representations*, In: Emergent Nature, (M. M. Novak, ed.), World Scientific, 2002, pp. 435–436.

[6] LJ. M. KOCIĆ, A. C. SIMONCELLI: *Towards free-form fractal modelling*, In: Mathematical Methods for Curves and Surfaces II, (M. Daehlen, T. Lyche and L. L. Schumaker, eds.), Vanderbilt University Press, Nashville (TN.), 1998, pp. 287–294.

[7] LJ. M. KOCIĆ, A. C. SIMONCELLI: *Stochastic approach to affine invariant IFS*, In: Prague Stochastics'98 (Proc. 6th Prague Symp., Aug. 23-28, M. Hruskova, P. Lachout and J.A. Visek eds), Vol II, Charles Univ. and Academy of Sciences of Czech Republic, Union of Czech Mathematicians and Physicists, Prague 1998, pp. 317–320.

[8] LJ. M. KOCIĆ, S. GEGOVSKA-ZAJKOVA, E. BABAČE : *Orthogonal decompostiton of fractal sets*, Approximation and Computation, Vol.42, (Gautschi, W., Mastroianni, G., Rassias, T. M., eds.), Series: Springer Optimization and Its Application, 2010, pp. 145–156.

[9] LJ. M. KOCIĆ, S. GEGOVSKA-ZAJKOVA, V. ANDOVA: *Interpolation polynomials via AIFS*, Bulletin mathématique de la société des mathématiciens de la République de Macédoine, 33(LIX), 2009, pp. 63–726.

[10] J. M. LANE, R. F. RIESENFELD: *A theoretical development for the computer generation and display of piecewise polynomial surfaces*, IEEE Transactions on Pattern Analysis and Machine Intelligence, PAMI-2(1):35 46, January 1980.