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Stability of Equilibrium Positions of Mechanical Systems with Switched Force Fields

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Abstract: Mechanical system with switched dissipative and potential forces is considered. It is assumed that dissipative forces are linear, whereas potential forces are nonlinear and homogeneous. The conditions of the existence of a common Lyapunov function for corresponding family of subsystems are investigated. The fulfilment of these conditions provides asymptotic stability of the equilibrium position of the switched system for any admissible switching law.

Keywords: mechanical systems, switched forces, asymptotic stability, common Lyapunov function

1 Introduction

Stability analysis and synthesis of switched systems are fundamental and challenging research topics of modern control theory [3, 4]. In various cases, it is required to design a control system in such a way that it remains stable for any admissible switching law [3]. A general approach to the above problem is based on the computation of a common Lyapunov function (CLF) for a family of subsystems corresponding to the switched system. This approach has been effectively used in many papers, see, for instance, [3, 4] and references therein. However, conditions of the existence of a CLF are not completely investigated even for family of linear time-invariant systems.

This problem is especially complicated for mechanical systems with switched force fields. Motions of such systems are described usually by differential equations of the second order. This results in the appearance of certain special properties. Therefore, well known approaches developed for systems of general form may be inefficient or even inapplicable for mechanical systems [1, 2]. Thus, the problem of the existence of CLFs for families of mechanical systems requires independent and advanced study.

In this paper, mechanical system with switched dissipative and potential forces is considered. It is assumed that dissipative forces are linear, whereas potential forces are nonlinear and homogeneous. The conditions of the existence of a CLF for corresponding family

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of subsystems are investigated. The fulfilment of these conditions provides asymptotic stability of the equilibrium position of the switched system for any admissible switching law.

2 Statement of the problem

Let the family of mechanical systems

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q}-\frac{\partial T}{\partial q}=-B_s\dot q-\frac{\partial \Pi_s(q)}{\partial q},\qquad s=1,\dots,N, \tag{1}$$

be given. Here q and $\dot q$ are n-dimensional vectors of generalized coordinates and generalized velocities respectively; the kinetic energy $T=T(q,\dot q)$ of the systems is of the form $T(q,\dot q)=\frac{1}{2}\dot q^T A(q)\dot q$, where $A(q)$ is a symmetric and continuously differentiable for $q\in R^n$ matrix; B_s are constant symmetric and positive definite matrices; potentials $\Pi_s(q)$ are continuously differentiable for $q\in R^n$ positive definite homogeneous of the order $\mu+1$ functions, $\mu>1$; $s=1,\dots,N$. Thus, we consider holonomic mechanical systems with linear dissipative forces and essentially nonlinear potential ones.

Assume that for the kinetic energy the estimates

$$k_1\|\dot q\|^2\le T(q,\dot q)\le k_2\|\dot q\|^2,\qquad \left\|\frac{\partial T(q,\dot q)}{\partial \dot q}\right\|\le k_3\|\dot q\|,\qquad \left\|\frac{\partial T(q,\dot q)}{\partial q}\right\|\le k_4\|\dot q\|^2$$

hold for all $q,\dot q\in R^n$, where k_1,k_2,k_3,k_4 are positive constants, and $\|\cdot\|$ is the Euclidean norm of a vector.

Switched system generated by the family (1) and a switching law σ is

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q}-\frac{\partial T}{\partial q}=-B_\sigma\dot q-\frac{\partial \Pi_\sigma(q)}{\partial q}. \tag{2}$$

In the present paper, a switching law is defined as a piecewise constant function $\sigma=\sigma(t):[0,+\infty)\to S=\{1,\dots,N\}$. We assume that on every bounded time interval the function has a finite number of discontinuities, which are called switching instants of time, and takes a constant value on every interval between two consecutive switching instants.

Switched system (2) and subsystems (1) possess the equilibrium position $q=\dot q=0$. For every subsystem from the family (1), the equilibrium position is asymptotically stable. Let us determine the conditions under which the equilibrium position $q=\dot q=0$ of switched system is asymptotically stable for any admissible switching law.

3 Conditions of the asymptotic stability

First, consider the case where switching takes place in the dissipative forces only.

Theorem 1. Let $\Pi_s(q)=\Pi(q)$, $s=1,\dots,N$, and $\Pi(q)$ is continuously differentiable for $q\in R^n$ positive definite homogeneous of the order $\mu+1$ function, $\mu>1$. Then the equilibrium position $q=\dot q=0$ of (2) is asymptotically stable for any admissible switching law.

Proof. In the considered case, family (1) can be rewritten as follows

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = -B_s\dot{q} - \frac{\partial \Pi(q)}{\partial q}, \qquad s = 1,\ldots,N. \tag{3}$$

Construct the CLF for family (3) in the form

$$V(q,\dot{q}) = T(q,\dot{q}) + \Pi(q) + \gamma \|q\|^{\mu-1} q^T \frac{\partial T}{\partial \dot{q}}, \qquad \gamma = \text{const} > 0.$$

Function $V(q,\dot{q})$ and its derivative with respect to the j-th subsystem from (3) satisfy the following estimates

$$k_1 \|\dot{q}\|^2 + p_1 \|q\|^{\mu+1} - k_3 \|\dot{q}\| \|q\|^{\mu} \le V(q,\dot{q}) \le k_2 \|\dot{q}\|^2 + p_2 \|q\|^{\mu+1} + k_3 \|\dot{q}\| \|q\|^{\mu},$$

$$\dot{V}\big|_{(j)} \le -b_{1j} \|\dot{q}\|^2 - \gamma(\mu+1)p_1 \|q\|^{2\mu} + \gamma b_{2j} \|\dot{q}\| \|q\|^{\mu} + \gamma k_4 \|\dot{q}\|^2 \|q\|^{\mu} + \gamma k_3 c \|\dot{q}\|^2 \|q\|^{\mu-1}$$

for all $q,\dot{q} \in R^n$ and $j = 1,\ldots,N$. Here $p_1, p_2, b_{1j}, b_{2j}, c$ are positive constants.

By the use of generalized homogeneous functions properties [5], it is easy to verify that, for sufficiently small values of γ, function $V(q,\dot{q})$ is positive definite, whereas its derivative with respect to every subsystem from (3) is negative defined function. That completes the proof.

Remark 1. Theorem 1 is well consistent with similar result known for the case of linear ($\mu = 1$) potential forces [1].

Next, assume that switched forces are potential ones only.

Theorem 2. Let $B_s = B$, $s = 1,\ldots,N$, and B is a constant symmetric and positive definite matrix. Then the equilibrium position $q = \dot{q} = 0$ of (2) is asymptotically stable for any admissible switching law.

Proof. Consider the corresponding family of subsystems

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = -B\dot{q} - \frac{\partial \Pi_s(q)}{\partial q}, \qquad s = 1,\ldots,N. \tag{4}$$

Choose the CLF for (4) in the form

$$V(q,\dot{q}) = hT(q,\dot{q}) + \frac{1}{2}q^T Bq + q^T \frac{\partial T}{\partial \dot{q}}, \qquad h = \text{const} > 0. \tag{5}$$

We obtain

$$hk_1 \|\dot{q}\|^2 + b_1 \|q\|^2 - k_3 \|\dot{q}\| \|q\| \le V(q,\dot{q}) \le hk_2 \|\dot{q}\|^2 + b_2 \|q\|^2 + k_3 \|\dot{q}\| \|q\|,$$

$$\dot{V}\big|_{(j)} \le -hb_1 \|\dot{q}\|^2 - (\mu+1)p_j \|q\|^{\mu+1} + 2k_2 \|\dot{q}\|^2 + k_4 \|\dot{q}\|^2 \|q\| + ha \|\dot{q}\| \|q\|^{\mu}$$

for $q,\dot{q} \in R^n$ and $j = 1,\ldots,N$. Here p_j, b_1, b_2, a are positive constants. Hence [5], for sufficiently large values of h, (5) is the CLF for family (4). That completes the proof.

Remark 2. It is known [2] that the statement of Theorem 2 is not true for systems with linear ($\mu = 1$) potential forces.

Finally, consider the general case where switching takes place both in the dissipative forces and in the potential ones.

For the each j-th subsystem from family (1), choose the Lyapunov function in the form

$$V_j(q,\dot q) = T(q,\dot q) + \Pi_j(q) + \gamma_j \|q\|^{\mu-1} q^T \frac{\partial T}{\partial \dot q}, \quad \gamma_j > 0, \quad j = 1,\dots,N. \tag{6}$$

Next, consider a convex combination of functions (6). Let

$$V(q,\dot q) = T(q,\dot q) + \sum_{i=1}^{N} \lambda_i \Pi_i(q) + \gamma \|q\|^{\mu-1} q^T \frac{\partial T}{\partial \dot q},$$

where $\lambda_i \geq 0$, $\lambda_1 + \ldots + \lambda_N = 1$, and $\gamma = \lambda_1\gamma_1 + \ldots + \lambda_N\gamma_N$. For any values of $\lambda_1,\dots,\lambda_N$, function $V(q,\dot q)$ is positive definite.

Let $\underline{b}_j$ and $\overline{b}_j$ be minimal and maximal eigenvalues of matrix B_j respectively, and

$$\Theta_{ij} = \max_{\|q\|=1} \left\| \frac{\partial \Pi_i(q)}{\partial q} - \frac{\partial \Pi_j(q)}{\partial q} \right\|, \quad c = \max_{\|q\|=1} \left\| \frac{\partial \left(\|q\|^{\mu-1} q \right)}{\partial q} \right\|,$$

$$p_j = (\mu+1) \min_{\|q\|=1} \Pi_j(q), \quad \omega_j = \sum_{s=1}^{N} \lambda_s \Theta_{sj}, \quad i,j = 1,\dots,N.$$

Differentiating $V(q,\dot q)$ along the solutions of the j-th subsystem from (1), we obtain that the estimate

$$\dot V\big|_{(j)} \leq -\underline{b}_j \|\dot q\|^2 - \gamma p_j \|q\|^{2\mu} + \left(\gamma \overline{b}_j + \omega_j\right) \|q\|^{\mu} \|\dot q\| + \gamma k_4 \|q\|^{\mu} \|\dot q\|^2 + \gamma c k_3 \|q\|^{\mu-1} \|\dot q\|^2$$

holds for all $q,\dot q \in R^n$, $j = 1,\dots,N$. Hence [5], if the quadratic forms

$$W_j(y_1,y_2) = -\underline{b}_j y_1^2 - \gamma p_j y_2^2 + \left(\gamma \overline{b}_j + \omega_j\right) y_1 y_2, \qquad j = 1,\dots,N,$$

are negative definite, then the functions $\dot V\big|_{(1)},\dots,\dot V\big|_{(N)}$ possess the same property.

Applying the Silvester criterion, we arrive at the conditions

$$\gamma \overline{b}_j + \omega_j < 2\sqrt{\underline{b}_j p_j}\sqrt{\gamma}, \qquad j = 1,\dots,N. \tag{7}$$

Denote $\eta_j = \sqrt{\underline{b}_j p_j} / \overline{b}_j$, $j = 1,\dots,N$. Then inequalities (7) can be rewritten as follows

$$\omega_j < \overline{b}_j \eta_j^2, \quad \eta_j - \sqrt{\eta_j^2 - \omega_j / \overline{b}_j} < \sqrt{\gamma} < \eta_j + \sqrt{\eta_j^2 - \omega_j / \overline{b}_j}, \quad j = 1,\dots,N.$$

Thus, the following theorem is valid.

Theorem 3. If there exist nonnegative numbers $\lambda_1,\dots,\lambda_N$ satisfying the conditions

$$\sum_{s=1}^{N} \lambda_s = 1, \qquad \sum_{s=1}^{N} \lambda_s \Theta_{sj} < \overline{b}_j \eta_j^2, \quad j = 1,\dots,N,$$

$$\max_{i=1,\dots,N}\left(\eta_i-\sqrt{\eta_i^2-\sum_{s=1}^{N}\lambda_s\Theta_{si}/\,\overline{b}_i}\right)<\min_{i=1,\dots,N}\left(\eta_i+\sqrt{\eta_i^2-\sum_{s=1}^{N}\lambda_s\Theta_{si}/\,\overline{b}_i}\right),$$

then the equilibrium position $q=\dot{q}=0$ of (2) is asymptotically stable for any admissible switching law.

Corollary 1. Let there exist nonnegative numbers $\lambda_1,\dots,\lambda_N$ such that

$$\sum_{s=1}^{N}\lambda_s=1,\qquad \sum_{s=1}^{N}\lambda_s\Theta_{sj}<\frac{4\beta\overline{b}_j\eta_j^2}{(1+\beta)^2},\quad j=1,\dots,N,$$

where $\beta=\min_{s=1,\dots,N}\ \eta_s/\max_{s=1,\dots,N}\ \eta_s$. Then the equilibrium position $q=\dot{q}=0$ of (2) is asymptotically stable for any admissible switching law.

Remark 3. It can be verified that the application of the similar approach to systems with linear potential forces ($\mu=1$) gives more severe constraints than in the case of nonlinear potentials. In particular, for linear systems, conditions of asymptotic stability depend on inertial characteristics of the system, whereas, for nonlinear ones, conditions of Theorem 3 and Corollary 1 do not depend of parameters k_1,k_2,k_3,k_4.

4 Conclusions

In the present paper, a class of nonlinear mechanical systems with switched force fields was investigated. The asymptotic stability conditions of the equilibrium position are obtained on the base of CLF constructing for corresponding family of subsystems. It is shown that, for essentially nonlinear subsystems, we can guarantee the existence of a CLF under weaker assumptions than in the linear case. Thus, in comparison with linear systems, nonlinear ones are “more stable” with respect to switching.

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