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Stochastic Stability Conditions for Solutions of Differential Equations for Means

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Abstract: An approach is proposed for the study of the solutions stability of differential equations for random Markov processes mean values, based on an analysis of the behavior of variance.

Keywords: Markov process, stability conditions, differential equations

1 Introduction

The Laplaces determinism theory is that any phenomenon could be described by some differential equation. The concepts of conditionality and stability of differential equations solutions, as well as probabilistic descriptions of different phenomena are based on Laplaces ideas. Lyapunov obtained the stability conditions of differential equations solutions. The first definition of *Lyapunov stability* for the Cauchy problem $y = \Phi(t,y), y(t_0) = y_0$ is as follows. If for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that if $|y_1(t_0) - y_1(t_0)| < \delta(\varepsilon)$ then $|y_1(t) - y_1(t)| < \delta(\varepsilon)$ for any $t > t_0$. *Asympotic stability* means, in addition, that there is a $\delta > 0$ such that if $|y_1(t_0) - y_1(t_0)| < \delta$ then $\lim\limits_{t\to\infty} |y_1(t) - y_1(t)| = 0$.

Most events are random in nature and are subject to external influences that can not be explained from deterministic point of view, that is why stochastic models based of probability density function (PDF) are promising. Literature review on stochastic models of thermal conductivity, diffusion, heat and mass transfer, electrodynamics, etc. could be found in [1].

2 Main results

This paper is devoted to the stability conditions for the Cauchy problem for means, which is based on Fokker – Planck – Kolmogorov equation. This equation for PDF (not taking jumps into account) looks as follows:

$$\partial\Pi(t,C)/\partial t = -\partial\left(f(t,C)\Pi(t,C)\right)/\partial C + B\partial^2\Pi(t,C)/2C^2, \tag{1}$$

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where t means time, C – concentration, $\Pi(t,C)$ – PDF, $f(t,C)$ – drift term, $B(t,C)$ – diffusion term.

Mean value of random process with PDF $\Pi(t,C)$ satisfying (1) is

$$M^{(1)}(t) = \int_0^{+\infty} C\Pi(t,C)dC,$$

second raw moment is

$$M^{(2)}(t) = \int_0^{+\infty} C^2\Pi(t,C)dC,$$

variance is

$$\sigma^2(t) = M^{(2)}(t) - \left(M^{(2)}(t)\right)^2.$$

Let us formulate the problems for PDF, mean value and second raw moment:

$$\begin{cases} \partial\Pi(t,C)/\partial t = -\partial\left(f(t,C)\Pi(t,C)\right)/\partial C + B\partial^2\Pi(t,C)/2C^2, \\ t > 0, C \in (-\infty,+\infty), \\ \Pi(t,C) = \Pi_0(t,C), C \in (-\infty,+\infty), \\ \Pi(t,0) = \Pi(t,+\infty) = \Pi(+\infty,C); \end{cases}$$

$$\begin{cases} \partial M^{(1)}(t)/\partial t = \int\limits_0^{+\infty} f(t,C)dC, t > 0, \\ M^{(1)}(0) = M_0^{(1)}; \end{cases} \tag{2}$$

$$\begin{cases} \partial M^{(2)}(t)/\partial t = 2\int\limits_0^{+\infty} Cf(t,C)dC + B, t > 0, \\ M^{(2)}(0) = M_0^{(2)} = \sigma_0^2 + \left(M_0^{(1)}\right)^2. \end{cases}$$

When $f(t,C) = f(t)$ the problem for second raw moment has the most simple form:

$$\begin{cases} \partial M^{(2)}(t)/\partial t = 2M^{(1)}(t) + B, t > 0, \\ M^{(2)}(0) = M_0^{(2)} = \sigma_0^2 + \left(M_0^{(1)}\right)^2. \end{cases}$$

A solution of (2) is called *stochastically stable* if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that if $\sigma_0^2 < \delta(\varepsilon)$ then $\sigma^2(t) < \varepsilon$ for any $t > t_0$.

A solution of (2) is called *stochastically asymptotically stable* when there is a $\delta > 0$ such that if $\sigma_0^2 < \delta$ then $\lim\limits_{t\to\infty} \sigma^2(t) = L = const < +\infty$.

If $\sigma_0^2 = L = 0$ then we will call the solution *stochastically absolutely stable* (in the latter case, we have an analog of the asymptotic Lyapunov stability [2]).

If $B = 0$ and $f(t) = kt$ the problem for mean value (3) takes the following form:

$$\begin{cases} \partial M^{(1)}(t)\big/\partial t = kt, t > 0, \\ M^{(1)}(0) = M_0^{(1)}, \end{cases}$$

and the solution of this problem is $M^{(1)}(t) = M_0^{(1)} \exp(kt)$.

When we know the mean value we can rewrite the problem for second raw moment:

$$\begin{cases} \partial M^{(2)}(t)\big/\partial t = 2ktM_0^{(1)} \exp(kt), t > 0, \\ M^{(2)}(0) = M_0^{(2)} = \sigma_0^2 + \left(M_0^{(1)}\right)^2, \end{cases}$$

and find its solution:

$$M^{(2)}(t) = 2kM_0^{(1)} \left(\left(t/k - 1/k^2\right) \exp(kt) + \sigma_0^2 + M_0^{(1)} \Big/ (2k) + 1/k^2 \right).$$

Then the variance is

$$\sigma^2(t) =$$

$$= 2kM_0^{(1)} \left(\left(t/k - 1/k^2\right) \exp(kt) + \sigma_0^2 + M_0^{(1)} \Big/ (2k) + 1/k^2 \right) - \left(M_0^{(1)}\right)^2 \exp(2kt).$$

If $k < 0$ then $\lim\limits_{t\to\infty} \sigma^2(t) = \sigma_0^2 + M_0^{(1)} \Big/ (2k) + 1/k^2 < +\infty$, and the solution is stochastically asymptotically stable.

If $k > 0$ then $\lim\limits_{t\to\infty} \sigma^2(t) = +\infty$, and the solution is not stochastically asymptotically stable.

Let us now investigate the case of $f(t) = kt$ and $B \neq 0$. In this case the problem for second raw moment

$$\begin{cases} \partial M^{(2)}(t)\big/\partial t = 2ktM_0^{(1)} \exp(kt) + B, t > 0, \\ M^{(2)}(0) = M_0^{(2)} = \sigma_0^2 + \left(M_0^{(1)}\right)^2, \end{cases}$$

has the following solution:

$$M^{(2)}(t) = 2kM_0^{(1)} \left(\left(t/k - 1/k^2\right) \exp(kt) + \sigma_0^2 + M_0^{(1)} \Big/ (2k) + 1/k^2 \right) + Bt,$$

thus the variance is

$$\sigma^2(t) = 2kM_0^{(1)} \left(\left(t/k - 1/k^2\right) \exp(kt) + \sigma_0^2 + M_0^{(1)} \Big/ (2k) + 1/k^2 \right) + Bt -$$

$$- \left(M_0^{(1)}\right)^2 \exp(2kt),$$

and therefore $\lim\limits_{t\to\infty} \sigma^2(t) = +\infty$ which means that the solution is not stochastically asymptotically stable.

3 Spatially distributed systems

For spatially distributed systems we have similar equations and similar results. If all the diffusion terms are equal to zero and all the real parts characteristic roots of the system of equations for variance are negative then the solution is stochastically asymptotically stable; if there is a characteristic root with positive real part then the solution is not stochastically asymptotically stable. If there is a non-zero diffusion term then the solution is also not stochastically asymptotically stable [3].

References

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