

Construction of the Second-Order Fuchsian Systems with Nilpotent Irreducible Residue Matrices

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Abstract: One inverse problem of analytic theory of linear differential equations is considered. Namely, the second-order Fuchsian systems with four critical points and nilpotent irreducible matrices-residua are constructed.

Keywords: Fuchsian system, nilpotent irreducible residue matrix, monodromy matrix, exponential monodromy matrix.

1 Introduction

Suppose $X = CP^1$ is a complex projective line, $a_j, j = 1, 2, 3, 4$ are arbitrary points from X , and $\bar{M} = \bigcup_{j=1}^4 a_j$. On the open set $M = \{CP^1 \setminus \bar{M}\}$ we observe Fuchsian system

$$dY = \omega Y, \quad (1)$$

where Y is a square matrix of order 2×2 , while ω is differential 1-form defined as

$$\omega = \sum_{j=1}^4 U_j d \ln \left(\frac{x - a_j}{x_0 - a_j} \right). \quad (2)$$

It assumed that matrices U_j , of order 2×2 , are not dependent on x . Matrices U_j , called matrix-residue, satisfy the following condition [4]

$$\sum_{j=1}^4 U_j = 0. \quad (3)$$

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Denote with $\pi_1(M, x_0)$ a fundamental group of complex analytical multiplicity M , where $x_0 \in M$. Let $\Phi(x)$, be a branch of fundamental solution of Fuchsian system (1) ($\Phi(x_0) = \Phi_0$), which is transformed into another branch $\Phi_{\gamma_j}(x)$ during analytical extension along loop γ_j , $\gamma_j \in \pi_1(M, x_0)$. Thereby, matrix functions $\phi(x)$ and $\Phi_{\gamma_j}(x)$ are connected by the equality $\Phi_{\gamma_j}(x) = \Phi(x)V_j$. Matrices V_j belong to a group $GL(2; C)$ of nonsingular complex matrices of order 2×2 . It is obvious that $V_j = \Phi_0^{-1}\Phi_{\gamma_j}(x_0)$.

Matrices V_j , $j = 1, 2, 3, 4$, called monodromy matrices, have to satisfy the following condition

$$\prod_{j=1}^4 V_j = E, \quad (4)$$

where E denotes unity matrix. Let us note that matrices V_j , that satisfy condition (4), generate a multiplicative group referred to as monodromy group (see [4]).

Related to V_j , $j = 1, 2, 3, 4$ matrices are exponential monodromy matrices W_j , $j = 1, 2, 3, 4$, whereby the following equality is valid

$$V_j = e^{2\pi i W_j}, \quad (5)$$

for $j = 1, 2, 3, 4$, where i is an imaginary unit.

It was proved (see for example [9]) that eigenvalues of exponential monodromy matrices and residue matrices of system (1) coincide. This fact will be used in this paper.

2 Statement of the problem

Let

$$\chi : \pi_1(M, x_0) \mapsto GL(2; C), \quad (6)$$

is a homomorphism called monodromy or monodromy representation of system (1).

The Riemann-Hilbert problem [5]: Let monodromy (6) is given. The question is whether there exist a system (1)-(3), for the given points a_1, a_2, a_3, a_4 whose fundamental solution matrix is realized by the homomorphism (6)?

According to the results presented in [6] it can be concluded that this problem always has a solution. However, in a general case, for arbitrary points a_1, a_2, \dots, a_n and arbitrary system (1) of order $m \geq 3$, there is a monodromy (6) for which does not exist Fuchsian system that solves stated problem.

According to the formulation of the problem, it can be concluded that proof of the existence of the system with given properties is not constructive. In this paper we give constructive solution of the stated problem, for the residue nilpotent irreducible residue matrices, whereby none of them can be reduced to diagonal form.

3 Preliminary results

Prilikom konstrukcije sistema oblika (1) nameće se uslov potpune (kompletne) integrabilnosti???? This means that differential 1-form in (2) has to satisfy the condition [2]

$$d\omega = \omega \wedge \omega, \quad (7)$$

where \wedge denotes an operator of outer difference of matrices.

When monodromy matrices V_j are noncommutative, residue matrices U_j are noncommutative as well. Analogously to the proof of Lemma 1 from [1], it can be proved that according to (7) the following equation is valid

$$dU_j = \sum_{\substack{k=1 \\ k \neq j}}^4 [U_k, U_j] d \ln \frac{a_j - a_k}{x_0 - a_k} \quad (8)$$

where $[\cdot, \cdot]$ denotes Li product of matrices (commutator). More details about the equality (8) can be found in [8, 9].

Without loss of generality, when choosing matrices $W_j, j = 1, 2, 3, 4$, three out of four will can be chosen to be nilpotent and of the form

$$W_j = \begin{bmatrix} -\mu_j v_j & \mu_j^2 \\ -v_j^2 & \mu_j v_j \end{bmatrix}, \quad j = 1, 2, 3, \quad (9)$$

where μ_j and v_j are real or complex numbers. These are chosen such that, according to condition (4), matrix W_4 is nilpotent as well.

From (9) it follows that eigenvalues of matrices $W_j, j = 1, 2, 3$ are $\xi_j^{(1)} = \xi_j^{(2)} = 0$. Therefore, eigenvalues of matrices $U_j, j = 1, 2, 3$ are equal to zero. As a consequence, residue matrix of system (1) has to be of the following form

$$U_j = \begin{bmatrix} -\eta_j \theta_j & \eta_j^2 \\ -\theta_j^2 & \eta_j \theta_j \end{bmatrix}, \quad j = 1, 2, 3 \quad (10)$$

where η_j, θ_j are some parameters.

Since $U_4 = -U_1 - U_2 - U_3$ and matrices $W_j, j = 1, 2, 3$ are noncommutative, eigenvalues of matrix U_4 have to be different from zero. However, from the condition that U_4 is nilpotent, according to (3) and (10), the following equality must be satisfied

$$\Delta_{12}^2 + \Delta_{13}^2 + \Delta_{23}^2 = 0, \quad (11)$$

where

$$\Delta_{12} = \eta_1 \theta_2 - \eta_2 \theta_1, \quad \Delta_{13} = \eta_1 \theta_3 - \eta_3 \theta_1, \quad \Delta_{23} = \eta_2 \theta_3 - \eta_3 \theta_2. \quad (12)$$

According to (11) it follows that matrices (9) have to be given in a form that enables that fixed branch of matrix

$$W_4 = \frac{1}{2\pi i} \ln V_4,$$

where $V_4 = V_3^{-1}V_2^{-1}V_1^{-1}$, has eigenvalues equal to zero. Here are examples of matrices $W_j, j = 1, 2, 3, 4$ with the specified features

$$W_1 = \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0 & \mu \\ 0 & 0 \end{pmatrix}, \quad W_3 = \begin{pmatrix} 0 & -\mu \\ 0 & 0 \end{pmatrix}, \quad W_4 = \begin{pmatrix} 0 & 0 \\ -v & 0 \end{pmatrix}$$

In the text that follows, without loss of generality, we will take $a_4 = \infty$. Now, system (1) can be written as

$$dY = \left(\sum_{j=1}^3 \frac{U_j}{x - a_j} \right) Y dx \quad (13)$$

It was shown in [7] that matrices $U_j, j = 1, 2, 3$ from (1) satisfy the following equalities

$$U_j^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad U_j U_k = \rho U_k = \rho_{jk} E - U_k U_j, \quad U_j U_k U_j = U_j \rho_{jk} \quad (14)$$

$$\sigma_5 = \rho_{123} + \rho_{132} = 0, \quad \sigma_4 = \rho_{123} - \rho_{132} = 2\rho_{123},$$

where $\rho_{jk} = \sigma(U_j U_k)$, $\rho_{123} = \sigma(U_1 U_2 U_3)$, $\rho_{132} = \sigma(U_1 U_3 U_2)$, while $\sigma(T)$ represents a sum of diagonal elements of matrix T (i.e. a trace of matrix T).

4 The N. P. Erugin system

In [7] an algorithm for constructive solution of the above mentioned problem was proposed. It is based on five ordinary differential equations which reflect the dependence of traces of product residue matrices of Fuchsian system on singularities. Before applying proposed algorithm it is necessary to solve a mentioned system of differential equations which we will call N. P. Erugin system.

In our case N. P. Erugin system boils down to the following system of differential equations

$$\frac{d\rho_{12}}{dz} = \frac{\sigma_4}{z}, \quad \frac{d\rho_{13}}{dz} = \frac{\sigma_4}{1-z}, \quad \frac{d\rho_{23}}{dz} = \frac{\sigma_4}{z(z-1)}, \quad \frac{d\sigma_4}{dz} = \frac{2\rho_{13}(\rho_{12} - \rho_{23})}{z} + \frac{2\rho_{12}(\rho_{23} - \rho_{13})}{z-1} \quad (15)$$

where $z = \frac{a_3 - a_1}{a_3 - a_2}$.

Let us note that system (15) can be obtained according to (8), if we apply a procedure used in proof of Theorem 1 in paper [3]. Note that, in some occasions, N. P. Erugin system can have a stationary solutions. Solution of this problem is fundamental for solving Riemann-Hilbert problem. The existence of stationary solutions simplifies solution of stated problem.

According to (15) and (12) we have that

$$\rho_{12} = -\Delta_{12}^2, \quad \rho_{13} = -\Delta_{13}^2, \quad \rho_{23} = -\Delta_{23}^2, \quad \sigma_4 = 2\Delta_{12}\Delta_{13}\Delta_{23}. \quad (16)$$

If matrices in (10) are commutative, the above equalities become

$$\Delta_{12} = \Delta_{13} = \Delta_{23} = 0, \quad (17)$$

which implies that N. P. Erugin system (15) has a unique solution

$$\rho_{12} = \rho_{13} = \rho_{23} = \sigma_4 = 0.$$

Theorem 1 *If residue matrices (10) in system (13) are not commutative and satisfy the condition (11), then system of differential equations (15) does not have a stationary solution.*

Proof. Suppose that system (15) has a non-stationary solution. In that case we have that $\sigma_4 = 0$. Then according to (16) at least one of Δ_{12} , Δ_{13} or Δ_{23} must be zero. Suppose that $\Delta_{23} = 0$. But, then $\rho_{23} = 0$, and according to (15) and (11) we have also $\rho_{12} = 0$ and $\rho_{13} = 0$. So the equality (17) is obtained, which implies that matrices in (10) are commutative. This is in contradiction with the condition of Theorem 1. ■

5 Non-stationary solutions of the N. P. Erugin system

Lemma 1 *System of the first three equations in (15) has a solution which can be represented by the following power series that are convergent in the area $|z| < 1$,*

$$\begin{aligned} \rho_{12} &= -A_0 + \sum_{n=1}^{+\infty} \left(\frac{n-1}{n} A_{n-1} - A_n \right) z^n, \\ \rho_{13} &= -\sum_{n=2}^{+\infty} \frac{n-1}{n} A_{n-1} z^n, \\ \rho_{23} &= \sum_{n=0}^{+\infty} A_n z^n \\ \sigma_4 &= \sum_{n=1}^{+\infty} ((n-1)A_{n-1} - nA_n) z^n, \end{aligned} \quad (18)$$

where A_k are complex numbers and $A_0 \neq 0$.

Proof. By the immediate check-out one can verify that power series defined in (18) represent, formally, a solution of the first three equations in system (15).

Concerning the statement that power series converge in the radius $|z| < 1$, it is enough to integral the first two equations in (15) from 0 to z , and then use the formulas of Cauchy-Hadamard for determining radius of convergence of power series. Also, note that according to (11) and (16), we have that

$$\rho_{23} = -\rho_{12} - \rho_{13}, \quad \sigma_4^2 = 4\rho_{12}\rho_{13}\rho_{23}.$$

Lemma 2 *System of differential equations (15) has non-stationary solutions that can be represented in the form of power series of type (18), that converge in the radius $|z| < 1$, if coefficients of these series satisfy the following recurrent relations*

$$\begin{aligned} A_n &= \frac{1}{n^2} \{ (n-1)[(2n-1)A_{n-1} - (n-2)A_{n-2} + \\ &+ 2[A_0(2\frac{n-2}{n-1}A_{n-2} - A_{n-1}) + A_1(2\frac{n-3}{n-2}A_{n-3} - A_{n-2}) + \dots \\ &+ A_{n-3}(2\frac{1}{2}A_1 - A_2) - A_{n-2}A_1 - A_{n-1}A_0] - [\frac{1}{2}A_1(2A_{n-2} - \frac{n-3}{n-2}A_{n-3}) + \\ &+ \dots + 2\frac{n-2}{n-1}A_{n-2}A_1 + 2\frac{n-1}{n}A_0] \}, \quad n = 1, 2, 3, \dots \end{aligned} \quad (19)$$

The proof of the Lemma is conducted by the method of undetermined coefficients, whereby we take into account that fourth equation in (15), after simplifying, can be represented in the following form

$$z(z-1)\frac{d\sigma_4}{dz} = 2(\rho_{13}(\rho_{23} - \rho_{12}) + \rho_{23}(\rho_{12} - \rho_{13})z).$$

6 Main result

In order to solve the problem stated in Section 2, we will look at the fundamental solution matrix of system (13) in the form of the following sequence (see [9])

$$\Phi_{x_0}(x) = E + \sum_{v=1}^{\infty} \sum_{j_1, \dots, j_v}^{(1,2,3)} L_{x_0}(a_{j_1}, \dots, a_{j_v} | x) U_{j_1} \dots U_{j_v}, \quad (20)$$

where $\sum_{j_1, \dots, j_v}^{(1,2,3)}$ contains 3^v elements which are obtained when indexes j_1, \dots, j_v , independently from each other, take values 1,2,3 with the coefficients

$$\begin{aligned} L_{x_0}(a_{j_1} | x) &= \int_{x_0}^x \frac{dx}{x - a_{j_1}} = \ln \left(\frac{x - a_{j_1}}{x_0 - a_{j_1}} \right), \dots, \\ L_{x_0}(a_{j_1}, a_{j_2}, \dots, a_{j_v} | x) &= \int_{x_0}^x \frac{L_{x_0}(a_{j_1}, a_{j_2}, \dots, a_{j_{v-1}} | x)}{x - a_{j_v}} dz. \end{aligned}$$

We form Residue matrices $U_j, j = 1, 2, 3$ in (13), so that the following inequality is valid (see [9])

$$V_j = E + \sum_{v=1}^{\infty} \sum_{j_1, \dots, j_v}^{(1,2,3)} P_j(a_{j_1}, \dots, a_{j_v} | x_0) U_{j_1} \dots U_{j_v}, \quad j = 1, 2, 3 \quad (21)$$

where

$$\begin{aligned} P_j(a_{j_1} | x_0) &= \begin{cases} 2\pi i, & \text{of } j = j_1 \\ 0, & \text{if } j \neq j_1 \end{cases}, \\ P_j(a_{j_1}, \dots, a_{j_v} | x_0) &= \frac{(2\pi i)^v}{v!}, \text{ if } j_1 = \dots = j_v = j, \\ P_j(a_{j_1}, \dots, a_{j_v} | x_0) &= \int_{a_j}^{x_0} \left(\frac{P_j(a_{j_1}, \dots, a_{j_{v-1}} | x_0)}{x_0 - a_{j_v}} - \frac{P_j(a_{j_2}, \dots, a_{j_v} | x_0)}{x_0 - a_{j_1}} \right) dx_0, \end{aligned}$$

where series in (21) is entire function coefficient matrices from (13).

Theorem 2 System (13), with nilpotent irreducible residue matrices that can not be diagonalized and satisfy condition (3), whose solution is given in the form defined by (20) and contains monodromy matrix (6), exists if $|z| < 1$.

Proof. As is known (see [9]), matrix (20) is normed in the point x_0 , which is defined in the fundamental solution matrix of system (13). Since matrices W_j are nilpotent, formulas (5) become

$$\begin{aligned} V_j &= e^{2\pi i W_j} = E + 2\pi i W_j, \quad j = 1, 2, 3, \\ V_{\infty} &= V_3^{-1} V_2^{-1} V_1^{-1} = e^{2\pi i W_{\infty}} = E + 2\pi i W_{\infty}. \end{aligned} \quad (22)$$

According to (22) it can be concluded that fundamental solution matrices $U_j, j = 1, 2, 3$ of system (13) will have monodromy (22), if the following is valid

$$\begin{aligned} \int_{\gamma_j} d\Phi_{x_0}(x) &= \Phi_{x_0}(x) \Big|_{x_0}^{x_0 e^{2\pi i}} = \Phi_{x_0}(x_0 e^{2\pi i}) - \Phi_{x_0}(x_0) = \\ &= \Phi_{x_0}(x_0) V_j - E = V_j - E = e^{2\pi i W_j} - E = 2\pi i W_j. \end{aligned} \quad (23)$$

According to (22) and (23) follows

$$W_j = U_j + \sum_{v=2}^{+\infty} \sum_{j_1, \dots, j_v}^{(1,2,3)} P_j^*(a_{j_1}, \dots, a_{j_v} | x_0) U_{j_1} \dots U_{j_v}, \quad (24)$$

where

$$P_j^*(a_{j_1}, \dots, a_{j_v} | x_0) = \frac{1}{2\pi i} P_j(a_{j_1}, \dots, a_{j_v} | x_0), \quad j = 1, 2, 3$$

Since eigenvalues of matrices $W_j, U_j, j = 1, 2, 3, W_\infty$ and U_∞ are equal to zero, then having in mind results obtained in [7], matrices $U_j, j = 1, 2, 3$ in system (13) should have the following form

$$U_j = a_j \overline{W}_1 + \beta_j \overline{W}_2 + \gamma_j [\overline{W}_1, \overline{W}_2], \quad (25)$$

where

$$\overline{W}_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \overline{W}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad [\overline{W}_1, \overline{W}_2] = \overline{W}_1 \overline{W}_2 - \overline{W}_2 \overline{W}_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

According to (24) and (14), it follows that trace $\sigma(U_j W_k)$, of product of matrices U_j and $W_k, j = 1, 2, 3$ and $k = 1, 2, 3$, can be represented in the form of the following series

$$\sigma(U_j W_k) = \sigma_{jk}(\rho_{12}, \rho_{13}, \rho_{23}, \rho_{123}) \quad (26)$$

over integer powers $\rho_{12}, \rho_{13}, \rho_{23}, \rho_{123}$, that converges for each finite value of arguments (see [7]).

Having in mind conditions given in Theorem 2, we can take

$$[W_1, W_2] \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (27)$$

By multiplying equality (25) with matrices W_1 and W_2 , defined by (9), from left side, the following system is obtained

$$\begin{cases} \sigma(U_j W_1) = \alpha_j \mu_1^2 - \beta_j \nu_1^2 + 2\gamma_j \mu_1 \nu_1, \\ \sigma(U_j W_2) = \alpha_j \mu_2^2 - \beta_j \nu_2^2 + 2\gamma_j \mu_2 \nu_2, \end{cases} \quad j = 1, 2, 3 \quad (28)$$

Left sides of equations (28) are series defined by (26).

Matrices U_j from (25) can be written as

$$U_j = \begin{pmatrix} -\gamma_j & \alpha_j \\ \beta_j & \gamma_j \end{pmatrix}, \quad j = 1, 2, 3 \quad (29)$$

Since eigenvalues of these matrices are zero, that they have to satisfy the following equations

$$\gamma_j^2 + \alpha_j \beta_j = 0, \quad j = 1, 2, 3 \quad (30)$$

Denote by $\sigma_{j1} = \sigma(U_j W_1)$ and $\sigma_{j2} = \sigma(U_j W_2)$. Then α_j, β_j and γ_j can be found from the following system of equations

$$\begin{cases} \mu_1^2 \alpha_j - \nu_1^2 \beta_j + 2\mu_1 \nu_1 \gamma_j = \sigma_{j1}, \\ \mu_2^2 \alpha_j - \nu_2^2 \beta_j + 2\mu_2 \nu_2 \gamma_j = \sigma_{j2}, \\ \alpha_j \beta_j + \gamma_j^2 = 0. \end{cases}$$

Thus we have

$$\gamma_j = \frac{1}{k_1^2}(\mu_2 v_2 \sigma_{j1} + \mu_1 v_1 \sigma_{j2} \pm k_2 \sqrt{\sigma_{j1} \sigma_{j2}}), \quad \alpha_j = \Delta_1 / \Delta, \quad \beta_j = \Delta_2 / \delta, \quad (31)$$

where $k_1 = \mu_1 v_2 - v_1 \mu_2$ and $k_2 = \mu_1 v_2 + v_1 \mu_2$. According to (27), k_1 and k_2 are not equal to zero, and the following is valid

$$\Delta = -k_1 k_2, \quad \Delta_1 = 2v_1 v_2 k_1 \gamma_j + \sigma_{j2} v_1^2 - \sigma_{j2} v_1^2, \quad \Delta_2 = -2\mu_1 \mu_2 k_1 \gamma_j + \sigma_{j2} \mu_1^2 - \sigma_{j2} \mu_1^2.$$

Finally, according to (31) we can determine elements of matrix defined in (29). From Lemma 1 and 2, eqn. (26), and equality $\rho_{123} = \frac{1}{2}\sigma_4$, it follows that fundamental matrix (20) which represents the solution of systems (13) and (3), has monodromies, defined by (6), on set M if $|z| < 1$. ■.

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