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New Proofs of Some Discrete Inequalities of Wirtinger's type

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Abstract: A new approach in proving some well known inequalities of Wirtinger' s type is presented in this paper. Proofs are short, elegant and based on one class of inequalities for real numbers

Keywords: Discrete inequalities, Wirtinger inequality.

1 Introduction

Let $x_0, x_1, \ldots,, \alpha_1, \alpha_2, \ldots$ and $\beta_1, \beta_2, \ldots$ are positive real numbers, whereby $\alpha_k \cdot \beta_k > 0$, for each $k \in N$. A classic inequality (for these numbers)

$$\left(\sqrt{\frac{\alpha_k}{\beta_k}} x_k \pm \sqrt{\frac{\beta_k}{\alpha_k}} x_{k-1} \right)^2 \geq 0 \tag{1}$$

holds for these numbers if and only if

$$\alpha_k x_k \pm \beta_k x_{k-1} = 0. \tag{2}$$

We will show that by appropriate choice of real numbers x_k, α_k and β_k in accordance with inequality (1), some discrete inequalities of Wirtinger's, i.e. Opial's type can be derived (see for example [7]). Let us note that these inequalities play an important role in many scientific and technical areas, such as Theory of differential and difference equations [1], Matrix theory [2], Geometry [10], etc.

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2 Main result

Case a)

Let $x_0, x_1, \ldots, x_{n+1}$ are arbitrary real numbers with property $x_0 = x_{n+1} = 0$. Having in mind inequality (1) we have that

$$-\sum_{k=1}^{n}\left(\frac{\alpha_k}{\beta_k}+\frac{\beta_{k+1}}{\alpha_{k+1}}\right)x_k^2 \;\le\; 2\sum_{k=1}^{n-1}x_k x_{k+1} \;\le\; \sum_{k=1}^{n}\left(\frac{\alpha_k}{\beta_k}+\frac{\beta_{k+1}}{\alpha_{k+1}}\right)x_k^2 .$$

If we substitute α_k and β_k in the above inequality with $\alpha_k = \sin(k-1)t$ and $\beta_k = \sin kt$, it becomes

$$-\cos t\sum_{k=1}^{n}x_k^2 \;\le\; \sum_{k=1}^{n-1}x_k x_{k+1} \;\le\; \cos t\sum_{k=1}^{n}x_k^2 . \tag{3}$$

Since for each $k, k \in N$, must be $\alpha_k\beta_k > 0$, parameter t has to satisfy the inequality $0 < t < \frac{\pi}{n}$. Therefore we can take $t = \frac{\pi}{n+1}$. Now, inequality (3) becomes

$$-\cos\frac{\pi}{n+1}\sum_{k=1}^{n}x_k^2 \;\le\; \sum_{k=1}^{n-1}x_k x_{k+1} \;\le\; \cos\frac{\pi}{n+1}\sum_{k=1}^{n}x_k^2 , \tag{4}$$

Equality on the left (right) side of inequality (4) holds if and only if $x_k = C\cdot(-1)^{k-1}\sin\frac{k\pi}{n+1}$, ($x_k = C\cdot\sin\frac{k\pi}{n+1}$), for $k = 1, 2, \ldots, n$ while $C > 0$ is an arbitrary constant.

Inequality (4) is discrete inequality of Opial's type and was proved in[7]. Left side of inequality (4) in a form of discrete inequality of Wirtinger's type was proved in [6], and right side in [4]. Let us note that proof of inequality (4) given in this proposal is simpler then those given in [4, 6, 7].

Case b)

Let $x_0, x_1, \ldots, x_n$ are arbitrary real numbers with property $x_0 = 0$. Then, according to (1) we have that

$$2\sum_{k=1}^{n-1}x_k x_{k+1} \ge -\sum_{k=1}^{n}\left(\frac{\alpha_k}{\beta_k}+\frac{\beta_{k+1}}{\alpha_{k+1}}\right)x_k^2 + \frac{\beta_{n+1}}{\alpha_{n+1}}x_n^2 , \tag{5}$$

and

$$2\sum_{k=1}^{n-1}x_k x_{k+1} \le \sum_{k=1}^{n}\left(\frac{\alpha_k}{\beta_k}+\frac{\beta_{k+1}}{\alpha_{k+1}}\right)x_k^2 - \frac{\beta_{n+1}}{\alpha_{n+1}}x_n^2 . \tag{6}$$

If in inequality (5) α_k and β_k are substituted by $\alpha_k = \sin(2k-2)t$ and $\beta_k = \sin 2kt$, then it becomes

$$2\sum_{k=1}^{n-1}x_k x_{k+1} \ge -2\cos 2t\sum_{k=1}^{n}x_k^2 - x_n^2 .$$

Since parameter t has to satisfy $0 < t < \frac{\pi}{2n}$, we take $t = \frac{\pi}{2n+1}$. Now, the above inequality becomes

$$2\sum_{k=1}^{n-1} x_k x_{k+1} \geq -2\cos\frac{2\pi}{2n+1}\sum_{k=1}^{n} x_k^2 - x_n^2.$$

i.e.

$$\sum_{k=1}^{n}(x_{k-1}-x_k)^2 \leq 2\left(1+\cos\frac{2\pi}{2n+1}\right)\sum_{k=1}^{n} x_k^2. \tag{7}$$

Equality in (7) holds if and only if $x_k = C\cdot(-1)^{k-1}\sin\frac{2k\pi}{2k+1}$, $k = 1,2,\ldots,n$ where $C > 0$ is an arbitrary constant. The inequality (7) was proved in [6] (see also [15]).

If in inequality (6) we take $\alpha_k = \cos(k-1)t$ and $\beta_k = \cos kt$, it becomes

$$2\sum_{k=1}^{n-1} x_k x_{k+1} \leq 2\cos t\sum_{k=1}^{n} x_k^2 + x_n^2.$$

Since parameter t has to satisfy $0 < t < \frac{\pi}{2n}$, we take $t = \frac{\pi}{2n+1}$. Now, the above inequality becomes

$$2\sum_{k=1}^{n-1} x_k x_{k+1} \leq 2\cos\frac{\pi}{2n+1}\sum_{k=1}^{n} x_k^2 + x_n^2.$$

i.e.

$$\sum_{k=1}^{n}(x_k-x_{k-1})^2 \geq 2\left(1-\cos\frac{\pi}{2n+1}\right)\sum_{k=1}^{n} x_k^2. \tag{8}$$

Equality in inequality (8) holds if and only if $x_k = C\cdot\sin\frac{k\pi}{2n+1}$, $k = 1,2,\ldots,n$ where $C > 0$ is an arbitrary constant.

Inequality (8) was proved in [4]) in a more complicated manner.

Case c)

Let $x_0, x_1, \ldots, x_n$ are arbitrary real numbers with property $x_0 = x_n = 0$. Then, according to (1) we have that

$$\sum_{k=1}^{n}\left(\frac{\alpha_k}{\beta_k}x_k^2 + \frac{\beta_k}{\alpha_k}x_{k-1}^2\right) \geq 2\sum_{k=1}^{n} x_k x_{k-1}.$$

If we replace α_k and β_k with $\alpha_k = \sin kt$ and $\beta_k = \sin(k+1)t$, the above inequality becomes

$$2\cos t\sum_{k=1}^{n} x_k^2 + \left(\frac{\sin 2t}{\sin t} - \frac{\sin(n+2)t}{(n+1)t}\right)x_0^2 \geq 2\sum_{k=1}^{n} x_k x_{k-1}. \tag{9}$$

By imposing the condition

$$\frac{\sin 2t}{\sin t} - \frac{\sin(n+2)t}{\sin(n+1)t} = 0,$$

we obtain that $t = \frac{\pi}{n}$. Now inequality (9) becomes

$$\cos\frac{\pi}{n}\sum_{k=1}^{n} x_k^2 \ \geq \ \sum_{k=1}^{n} x_k x_{k-1} \tag{10}$$

The inequality (10) has application in geometry, i.e. with convex polygons (see for example [3, 5, 9, 11, 12, 13, 14]. Namely if in (10) we perform the following substitutions with $x_k := x_k \cos\sigma_k$ and $x_k := x_k \sin\sigma_k$, $\sigma_0 = \sigma_n$, and then sum up the obtained inequalities, we obtain

$$\cos\frac{\pi}{n}\sum_{k=1}^{n} x_k^2 \ \geq \ \sum_{k=1}^{n} x_k x_{k-1}\cos(\sigma_k - \sigma_{k-1}).$$

After performing substitutions $\sigma_k - \sigma_{k-1} = \gamma_{k-1}$, for $k = 1,2,\ldots,n-1$, and $\sigma_n - \sigma_{n-1} = \gamma_{n-1} - \pi$, the above inequality transforms into

$$\cos\frac{\pi}{n}\sum_{k=1}^{n} x_k^2 \ \geq \ \sum_{k=1}^{n} x_k x_{k-1}\cos\gamma_{k-1}, \tag{11}$$

where $\gamma_0 + \gamma_1 + \ldots + \gamma_{n-1} = \pi$. The inequality (11) has been proved in [11].

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