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Deformed exponentials, operators and modeling population growth

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Abstract: In this paper we present a one–parameter deformation of exponential function and appropriate mathematical tools, such as deformed addition or deformed differential operators. In particular, we focus on the differential and difference properties of the introduced functions and related operators. Based on this we offer new model of population growth.

Keywords: Deformed exponential function, Differential operator, Population growth models.

1 Introduction

In the recent development of science, solving concrete real problems leaded to various generalizations and deformations of exponential function. One–parameter deformations have been proposed in the context of non-extensive statistic mechanics [1, 2], relativistic statistical theory [3] and quantum–group theory [4]. In [5] a variant of the deformed exponential function of two variables is introduced using a formal mathematical approach and mentioned generalizations and deformations can be viewed as the its special cases. This function and the corresponding operators make the environment for the study of new classes of polynomials [6] or generalized polynomials [7]. On the other hand, some generalizations of exponential function are presented in [8] in order to find applications in population dynamics [9, 10].

The paper is organized as follows. After section devoted to introduction, we present the deformed exponential function of two variables in the second section. In the third section, we observe some difference and differential operators and examine their relationship to the presented function. Finally, in the last section, we focus on its application in growth models in the population dynamics.

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2 The deformed exponential functions

At the start we will present a deformation of exponential function of two variables depending of a parameter $h \in \mathbb{R} \setminus \{0\}$, which is introduced in [5].

Let $I = (-\infty, -1/h)$ for $h < 0$ or $I = (-1/h, +\infty)$ for $h > 0$. The deformed exponential function $(x,y) \mapsto e_h(x,y)$ is defined by

$$e_h(x,y) = (1+hx)^{y/h} \qquad (x \in I,\ y \in \mathbb{R}). \tag{1}$$

It is obvious that

$$\lim_{h\to 0} e_h(x,y) = e^{xy}\ .$$

If $h = 1-q$ $(q \neq 1)$ and $y = 1$, the function (1) becomes

$$e_{1-q}(x,1) = \big(1+(1-q)x\big)^{1/(1-q)}\ ,$$

i.e., $e_{1-q}(x,1) = e_q^x$, where e_q^x is Tsallis q–exponential function [1] defined by

$$e_q^x = \begin{cases} \big(1+(1-q)x\big)^{1/(1-q)}\ , & 1+(1-q)x > 0\ , \\ 0\ , & \text{otherwise}, \end{cases} \qquad (x \in \mathbb{R}).$$

If $h = p-1$ $(p \neq 1)$ and $x = 1$, the function (1) becomes

$$e_{p-1}(1,y) = p^{y/(p-1)},$$

i.e. function considered for a generalization of the classical exponential function in the context of quantum group formalism [11].

Notice that function (1) can be written in the form

$$e_h(x,y) = \exp\Big(\frac{y}{h}\ \ln(1+hx)\Big).$$

Hence, similar as in [12], we can use cylinder transformation as deformation function $x \mapsto \{x\}_h$ by

$$\{x\}_h = \frac{1}{h}\ \ln(1+hx) = \ln(1+hx)^{1/h} \qquad (x \in I)\ . \tag{2}$$

Thus, the following holds:

$$e_h(x,y) = e^{\{x\}_h\, y}\ . \tag{3}$$

We can show that the function (1) holds on some basic properties of exponential function.

Proposition 2.1 *For $x \in I$ and $y, y_1, y_2 \in \mathbb{R}$ the following holds:*

$$e_h(x,y) > 0, \qquad e_h(0,y) = e_h(x,0) = 1,$$
$$e_{-h}(x,y) = e_h(-x,-y), \qquad e_h(x,y_1+y_2) = e_h(x,y_1)e_h(x,y_2).$$

Notice that the additional property is true with respect to the second variable only. However, with respect to the first variable, the following holds:

$$e_h(x_1,y)e_h(x_2,y) = e_h\big(x_1+x_2+hx_1x_2,y\big)\ .$$

This equality suggests us to introduce a generalization of the sum operation

$$x_1 \oplus_h x_2 = x_1+x_2+hx_1x_2\ . \tag{4}$$

Such generalized addition operator was considered in some papers and books (see, for example, [2] or [12]). This operation is commutative, associative and zero is its neutral. For $x \neq -1/h$, the $\ominus_h$–inverse exists as

$$\ominus_h x = \frac{-x}{1+hx}$$

and $x \oplus_h (\ominus_h x) = 0$ is valid. Hence, $(I,\oplus_h)$ is an abelian group. In this way, the $\ominus_h$–subtraction can be defined by

$$x_1 \ominus_h x_2 = x_1 \oplus_h (\ominus_h x_2) = \frac{x_1-x_2}{1+hx_2} \qquad \Big(x_2 \neq -\frac{1}{h}\Big). \tag{5}$$

With respect to (2), we can prove the next equality for $x_1,x_2 \in I$:

$$\{x_1\}_h + \{x_2\}_h = \{x_1 \oplus_h x_2\}_h. \tag{6}$$

Really,

$$\begin{aligned}\{x_1\}_h + \{x_2\}_h &= \frac{1}{h}\ln\big(1+h(x_1+x_2+hx_1x_2)\big) = \frac{1}{h}\ln\big(1+h(x_1\oplus_h x_2)\big)\\ &= \{x_1 \oplus_h x_2\}_h.\end{aligned}$$

Proposition 2.2 *For $x_1,x_2 \in I$ and $y \in \mathbb{R}$, the following is valid:*

$$\begin{aligned}e_h(x_1 \oplus_h x_2, y) &= e_h(x_1,y)e_h(x_2,y),\\ e_h(x_1 \ominus_h x_2, y) &= e_h(x_1,y)e_h(x_2,-y).\end{aligned}$$

In order to find the expansions of the deformed exponential function, we introduce generalized backward integer power given by

$$z^{(0,h)} = 1, \qquad z^{(n,h)} = \prod_{k=0}^{n-1}(z-kh) \quad (n \in \mathbb{N},\ h \in \mathbb{R}\setminus\{0\}).$$

Proposition 2.3 *For function $(x,y) \mapsto e_h(x,y)$, the following representations hold:*

$$e_h(x,y) = \sum_{n=0}^{\infty}\frac{\{x\}_h^n y^n}{n!}\,, \qquad e_h(x,y) = \sum_{n=0}^{\infty}\frac{x^n y^{(n,h)}}{n!} \qquad (|hx|<1)\,. \tag{7}$$

Proof. The first representation can be obtained from (3) and the expansion of exponential function. The second one is based on expansion

$$(1+hx)^{y/h} = \sum_{n=0}^{\infty} \binom{y/h}{n} h^n x^n \quad (|hx|<1,\ y\in\mathbb{R})$$

and the relation

$$\binom{y/h}{n} = \frac{y(y-h)\cdots\big(y-(n-1)h\big)}{h^n\, n!} = \frac{y^{(n,h)}}{h^n\, n!}\ . \ \square \tag{8}$$

Following the generalization of exponential function $x \mapsto e^x$ to Laguerre–type exponential (or, shortly, L–exponential) functions

$$x \mapsto e_k(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^k} \qquad (k=1,2,\ldots), \tag{9}$$

presented in [8] and explored in [9, 10], we can define Laguerre–type deformed exponential or deformed L–exponential function $(x,y)\mapsto e_h^L(x,y)$ as

$$e_h^L(x,y) = \sum_{n=0}^{\infty} \frac{\{x\}_h^n y^n}{(n!)^2} \qquad (x\in I,\ y\in\mathbb{R}). \tag{10}$$

3 The deformed difference and differential operators

The introduced functions have some eigenvalue properties with respect to a few difference and differential operators, such like the exponential function has corresponding property with respect to the ordinary derivative D.

Firstly, let us recall that h–difference operator is

$$\Delta_{z,h} f(z) = \frac{f(z+h)-f(z)}{h}\ .$$

Further, deformed h–differential and h–derivative according to operation (4) is defined by means of (see [13])

$$d_h z = \lim_{u\to z} z \ominus_h u, \qquad D_{z,h} f(z) = \frac{df(z)}{d_h z} = \lim_{u\to z} \frac{f(z)-f(u)}{z\ominus_h u}\ .$$

With respect to (5) we have

$$D_{z,h} f(z) = \frac{df(z)}{d_h z} = \lim_{u\to z} \frac{f(z)-f(u)}{\dfrac{z-u}{1+hu}} = (1+hz)\frac{df(z)}{dz}\ . \tag{11}$$

The h–derivative holds on the property of linearity and the product rule:

$$D_{z,h}\big(\alpha f(z)+\beta g(z)\big) = \alpha D_{z,h} f(z) + \beta D_{z,h} g(z),$$
$$D_{z,h}\big(f(z)g(z)\big) = f(z) D_{z,h} g(z) + g(z) D_{z,h} f(z).$$

It is worth to notice that the following equality is valid:

$$D_{x,h}\left(\{x\}_h^n\right) = n\{x\}_h^{n-1} \qquad (n \in \mathbb{N}_0)\,. \tag{12}$$

For functions defined on I we define the operator

$$D_{z,h}^{-1} f(z) = \int_0^z \frac{f(t)}{1+ht}\, dt, \tag{13}$$

which is, in some sense, the inverse operator to operator $D_{z,h}$. It is easy to prove that

$$D_{x,h}^{-n} f(x) = \left(D_{x,h}^{-1}\right)^n f(x) = \frac{1}{(n-1)!}\int_0^x \frac{\left(\{x\}_h - \{t\}_h\right)^{n-1}}{1+ht} f(t)\, dt\,.$$

Also, let us noting that

$$D_{x,h}^{-1}\left(\{x\}_h^n\right) = \frac{\{x\}_h^{n+1}}{n+1} \qquad (n \in \mathbb{N}_0)\,. \tag{14}$$

It would be useful for further work to denote a simple multiplicative operator defined by

$$X_h f(x) = \{x\}_h f(x) = \ln(1+hx)^{1/h} f(x). \tag{15}$$

Finally, we define the deformed Laguerre derivative by

$$\begin{aligned} D_{x,h}^{L} f(x) &= \left(D_{x,h} X_h D_{x,h}\right) f(x) = \left(\frac{d}{d_h x}\{x\}_h \frac{d}{d_h x}\right) f(x) \\ &= (1+hx)\frac{d}{dx}\Big(\ln(1+hx)^{1/h}(1+hx)\frac{df(x)}{dx}\Big). \end{aligned}$$

This operator generalizes the Laguerre derivative DxD [8], which appears in mathematical modelling of some phenomena in viscous fluids and the oscillating chain in mechanics. Also, Laguerre derivative is used for modelling in population dynamic [10, 9], what encouraged the authors to offer some new models of population growth.

The eigenvalue properties of mentioned functions and operators are given by the next statements.

Proposition 3.1 [5] *The function $y \mapsto e_h(x,y)$ is the eigenfunction of difference operator $\Delta_{y,h}$ with eigenvalue x, i.e. the following holds:*

$$\Delta_{y,h}\, e_h(x,y) = x\, e_h(x,y).$$

Proposition 3.2 [5] *The function $e_h(x,y)$ is the eigenfunction of the operators $D_{x,h}$ and $\frac{\partial}{\partial y}$ with eigenvalues y and $\{x\}_h$ respectively, i.e.:*

$$D_{x,h}\, e_h(x,y) = y\, e_h(x,y), \qquad \frac{\partial}{\partial y} e_h(x,y) = \{x\}_h\, e_h(x,y).$$

Proposition 3.3 *The function $x \mapsto e_h^L(x,y)$ is the eigenfunction of the operator $D_{x,h}^L$ with eigenvalue y, i.e. the following holds:*

$$D_{x,h}^L\, e_h^L(x,y) = y\, e_h^L(x,y).$$

Proof. From the definitions of the deformed *L*–exponential function and equality (12) we have

$$\begin{aligned} D_{x,h}^L\, e_h^L(x,y) &= (D_{x,h}X_hD_{x,h})\left(\sum_{n=0}^{\infty}\frac{\{x\}_h^n y^n}{(n!)^2}\right) = D_{x,h}\left(\{x\}_h\sum_{n=1}^{\infty}\frac{n\{x\}_h^{n-1}y^n}{(n!)^2}\right)\\ &= \sum_{n=1}^{\infty}\frac{n^2\{x\}_h^{n-1}y^n}{(n!)^2} = \sum_{n=0}^{\infty}\frac{\{x\}_h^n y^n}{(n!)^2}. \ \square \end{aligned}$$

In the sequel we will give some differential properties of studied functions and operators [7].

Theorem 3.1 *For $x \in I$, $y \in \mathbb{R}$ and $k,n \in \mathbb{N}_0$ the following holds:*

$$D_{x,h}^k\big(\{x\}_h^n\big) = k!\binom{n}{k}\{x\}_h^{n-k}, \tag{16}$$

$$D_{x,h}^{-k}(1) = \frac{\{x\}_h^k}{k!}, \qquad \left(1 - yD_{x,h}^{-1}\right)^{-1}(1) = e_h(x,y).$$

Proof. The first two equalities follows from the definitions (2), (11) and (13). For third one, we recall the expansion (7) and the formal geometric series:

$$\begin{aligned} e_h(x,y) &= \sum_{n=0}^{\infty}\frac{\{x\}_h^n y^n}{n!} = \sum_{n=0}^{\infty} y^n D_{x,h}^{-n}(1) = \sum_{n=0}^{\infty}\left(yD_{x,h}^{-1}\right)^n(1)\\ &= \left(1 - yD_{x,h}^{-1}\right)^{-1}(1). \quad \square \end{aligned}$$

Theorem 3.2 *For $n \in \mathbb{N}_0$ the following is valid:*

$$(D_{x,h}X_hD_{x,h})^n = D_{x,h}^n X_h^n D_{x,h}^n\,.$$

Proof. The statement can be proven by mathematical induction with some suitable manipulations. $\square$

Theorem 3.3 *For $x \in I$, $y \in \mathbb{R}$ and $k,n \in \mathbb{N}_0$, the following is valid:*

$$(D_{x,h}X_hD_{x,h})\, e_h(x,y) = y\big(1 + yX_h\big)e_h(x,y)\,, \tag{17}$$

$$(D_{x,h}X_hD_{x,h})^k\left(\frac{\{x\}_h^n}{n!}\right) = k!\binom{n}{k}\frac{\{x\}_h^{n-k}}{(n-k)!}\,. \tag{18}$$

Proof. With respect to Proposition 3.2, equality (16) and the product rule for $D_{x,h}$, we have

$$(D_{x,h}X_hD_{x,h})\,e_h(x,y) = D_{x,h}\big(\{x\}_h y e_h(x,y)\big) = y\big(1+y\{x\}_h\big)e_h(x,y),$$

wherefrom we get the operational inscription. The second equality follows from repeated application of (16). □

At last, we refer to the M and P operators as the descending (or lowering) and ascending (or raising) operators associated with the polynomial set $\{q_n\}_{n\in\mathbb{N}_0}$ if

$$M(q_n) = nq_{n-1}\,, \qquad P(q_n) = q_{n+1}\,.$$

Then, the polynomial set $\{q_n\}_{n\in\mathbb{N}_0}$ is called quasi-monomial with respect to the operators M and P (see [8, 10]).

Considering (12) and (15), it is easy to see that $D_{x,h}$ and X_h are the descending and ascending operators associated with the set of generalized monomial $\{x\}_h^n$ $(n\in\mathbb{N}_0)$. Also, with respect to (18) and (14), $D^L_{x,h} = D_{x,h}X_hD_{x,h}$ and $D^{-1}_{x,h}$ are the descending and ascending operators associated with the set of generalized monomial $\dfrac{\{x\}_h^n}{n!}$ $(n\in\mathbb{N}_0)$.

4 Modelling population growth

In this section, we will note the presence and the potential use of the deformed exponentials in the growth models in the frameworks of population dynamics.

Let us consider the number $N(t)$ of population individuals at the time t with initial value $N(0)=N_0$. The model assumes that the increment of population in time period δt is proportional to $N(t)$, i.e. the following difference equation is satisfied

$$\Delta_{t,\delta t}N(t) = rN(t),$$

where r is called the *intrinsic growth rate*. According to Proposition 3.1, the function $t\mapsto e_{\delta t}(r,t)$ is an eigenfunction of the difference operator $\Delta_{t,\delta t}$ with eigenvalue r. Hence the solution of this equation can be expressed by the deformed exponential function:

$$N(t) = N_0 e_{\delta t}(r,t).$$

When $\delta t\to 0$, we get the Malthus model in population dynamics described by the equation

$$\frac{d}{dt}N(t) = rN(t), \qquad N(0)=N_0,$$

with the solution

$$N(t) = N_0e^{rt}\,.$$

In [9, 10], Laguerre–type derivative $D_t tD_t$ is used instead the ordinary derivative D_t. The obtained equation,

$$\frac{d}{dt}t\frac{d}{dt}N(t) = rN(t), \qquad N(0)=N_0,\ N'(0)=N_1=rN_0,$$

describes the L–Malthus model. In this case, the population growth increases according to the L–exponential function defined by (9):

$$N(t) = N_0 e_1(rt) = \sum_{n=0}^{\infty} r^n \frac{t^n}{(n!)^2} .$$

Thus, the relevant increase is slower with respect to the classical Malthus model.

If we substitute in classical Malthus model D_t with a deformed derivative $D_{h,t}$, we get deformed Malthus model described by the following equation:

$$\frac{d}{d_h t} N(t) = rN(t), \quad N(0) = N_0 .$$

According to Proposition 3.2, its solution is the deformed exponential function:

$$N(t) = N_0 e_h(t,r) = N_0(1+ht)^{r/h}.$$

With an appropriate choice of the constant $h > 0$, we can obtain an arbitrary level of the population growth increase.

Example. We will test our considerations on the world population $N(t)$ from the year 1965. till 2010. with data collected during 5 years periods, given in Table 1.

Table 1: THE WORLD POPULATION

Year	1965	1970	1975	1980	1985
N(t) (Billions)	3.346	3.708	4.087	4.454	4.850
Year	1990	1995	2000	2005	2010
N(t) (Billions)	5.276	5.686	6.079	6.449	6.870

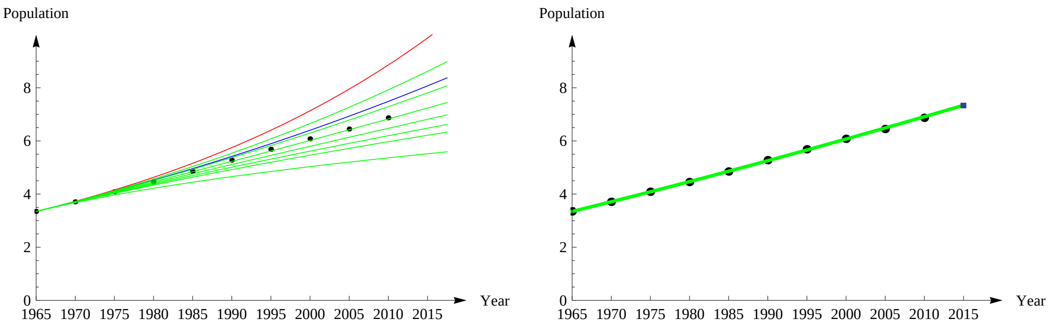

Figure 1: The solutions of Malthus, L–Malthus and deformed Malthus models of growth of world population

Let us compare the results obtained by deformed Malthus model with those obtained by classical and L–Malthus model given in [9]. In the left part of Figure 1, the upper bold–emphasized function is $N(t) = N_0 e^{rt}$, approximation provided by the Malthus model, and the lower is $N(t) = N_0 e_1(rt)$ provided by the L–Malthus model, without trial to do any prognosis for the future. The other curves are graphics of possible solutions of deformed Malthus model $N(t) = N_0 e_h(t,r)$ for different values of parameter $h = 0(0.015)0.15$.

In our model, we estimate h by the data from 1975. and we got $h_{\text{optimum}} = 0.0838058$. Its graphics and exact data were shown on the right part of Figure 1. Here, the maximal relative error is 0.6%. We are able to give prognosis that at the end of 2013. there will be 7 169 359 000 people, and in 2015. world population will be 7 341 754 000.

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