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Averaging of Evolution System with Boundary Conditions

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Abstract: Method of averaging was justified for normal differential equations with a large parameter and boundary conditions on a segment and on the half-axis. Illustrative examples are given in both cases.

Keywords: Boundary value problem, high frequency, method of averaging

1 Introduction

Method of averaging (see [1],[2]), which is usually associated with the names N. M. Krylov, N. N. Bogolyubov and Y. A. Mitropolsky, is one of the most important methods of asymptotic. In this paper, it is based for normal systems of ordinary differential equations with high non-linear part and the boundary conditions. Problem is considered on a finite time interval in section 1, problem is considered on a the positive half-axis in section 2. There is a material requirement in this paper: Jacobian matrix of the nonlinear part of the averaged problem is equal to 0 on solving this problem averaged. (For example, the mean f (1) on the last argument is zero.) We hope to remove this limitation in the next paper.

2 Problem on a segment

We consider the boundary value problem into time interval $t \in [0,1]$

$$\begin{cases} \dfrac{dx}{dt} = Ax + f(x,t,\omega t) \quad \omega \gg 1 \\ Lx(0) = l \\ Rx(1) = r. \end{cases} \tag{1}$$

A is a square matrix of order n, L and R are matrices, that are n columns, and k_L and k_R rows, l and r are k_L and k_R dimensional vector column.

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Define the norm of a vector and the norm of the matrix so that they are consistent:

$$|Ax| \le ||A|||x|.$$

We make the following assumptions about the data of the problem (1).

1. Let $k_L + k_R = n$ and determinant $\begin{vmatrix} L \\ R\exp A \end{vmatrix} \ne 0$ (see [3]).

2. Vector function $f(x,t,\tau)$ with values in $\mathbb{R}^n$ defined, continuous and continuously differentiable respect to the variable x on the set $S_\infty \equiv S \times [0;1] \times [0;\infty)$, where S is domain in $\mathbb{R}^n$. Moreover, $f(x,t,\tau)$ is limited and satisfies uniform the Holder condition respect to the variable t, that is, for any (x,t_1,τ), $(x,t_2,\tau) \in S_\infty$ performed inequalities
$$|f(x,t_1,\tau)| \le M_0,$$
$$|f(x,t_2,\tau) - f(x,t_1,\tau)| \le C|t_2 - t_1|^\gamma,$$
where $M_0 > 0, C > 0, \gamma \in (0;1)$ is constants that do not depend on x, t_1, t_2.

3. Jacobian matrix $f'_x(x,t,\tau)$ is limited and satisfies the Lipschitz condition uniformly in
$$(x,t,\tau) \in S_\infty$$
respect to the variable x, that is, there are positive constants M_1, λ, such that for any (x,t,τ), $(y,t,\tau) \in S_\infty$ have the estimates:
$$||f'_x(x,t,\tau)|| \le M_1,$$
$$||f'_x(x,t,\tau) - f'_x(y,t,\tau)|| \le \lambda|x-y|.$$

4. Uniformly in $(x,t) \in S \times [0;1]$ there are limits:
$$F(x,t) \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T f(x,t,\tau)\,d\tau, \quad F'_x(x,t) \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T f'_x(x,t,\tau)\,d\tau.$$

Along with the perturbed problem (1) consider the averaged problem
$$\begin{cases} \dfrac{dy}{dt} = Ay + F(y,t) \\ Ly(0) = l \\ Ry(1) = r \end{cases} \qquad (2)$$

5. Assume, that the problem (2) has a solution $\mathring{y}(t)$ and at the same $F'_y(\mathring{y}(t),t) = 0$.

We recall the definitions of known Banach spaces:$C([0;1])$ is space of continuous vector functions $u:[0;1]\to\mathbb{R}^n$ with max-norm. $C^{\mu}([0;1]),\mu\in(0,1)$ is space of vector functions $u(t)\in C([0;1])$ satisfying the condition Holder with index μ:

$$||u||_{C^{\mu}([0;1])}=||u||_{C([0;1])}+\sup_{0\le t_1<t_2\le 1}\frac{|u(t_2)-u(t_1)|}{(t_2-t_1)^{\mu}}<\infty.$$

Remark 1 *If $\overset{\circ}{y}(t)$ is a stationary solution of the problem (2) and the nonlinear part of the problem is independent of time, then with the help of the transition from matrix A to matrix $A+F'_y(\overset{\circ}{y})$ requirement $F'_y(\overset{\circ}{y},t)=0$ can be removed.*

Theorem 1 *Let the conditions 1 - 5 are satisfied. Then there exists $\omega_0>0$ such that in some $C^{\mu}([0;1])$ - neighborhood of vector function $\overset{\circ}{y}$ the problem (1) for $\omega>\omega_0$ has a unique solution x_ω and the following equality holds:*

$$\lim_{\omega\to\infty}||x_\omega-\overset{\circ}{y}||_{C^{\mu}([0;1])}=0.$$

3 Problem on the semi axis

We consider the problem on the semi axis $t\ge 0$

$$\begin{cases}\dfrac{dx}{dt}=Ax+f(x,t,\omega t) \quad \omega\gg 1\\ \qquad Mx(0)=\varphi.\end{cases} \tag{3}$$

A is a square matrix of order n, M is rectangular matrix, whose rows are linearly independent, φ is vector, whose dimension coincides with the number k rows of the matrix M.

We assume the following.

1. Matrix A has no purely imaginary eigenvalues.

2. For the problem $\dfrac{dv}{dt}=Av$ with the boundary condition $Mv(0)=0$ is satisfied Lopatinskii condition [3]. This means that the number of rows of the matrix M is equal to k and only limited on the positive half-by the decision of the boundary problem is the zero solution.

3. Vector function $f(x,t,\tau)$ with values in $\mathbb{R}^n$ defined, continuous and continuously differentiable respect to the variable x on the set $S_\infty\equiv S\times[0;\infty)\times[0;\infty)$, where S is domain in $\mathbb{R}^n$. Moreover, $f(x,t,\tau)$ is limited and satisfies uniform the Holder condition respect to the variable t, that is, for any (x,t_1,τ), $(x,t_2,\tau)\in S_\infty$ performed inequalities
$$|f(x,t_1,\tau)|\le K_0,$$
$$|f(x,t_2,\tau)-f(x,t_1,\tau)|\le C|t_2-t_1|^{\gamma},$$
$K_0,C>0$, $\gamma\in(0;1)$ is constants that do not depend on x, t_1, t_2.

4. Jacobian matrix $f'_x(x,t,\tau)$ is limited and satisfies the Lipschitz condition uniformly in

$$(x,t,\tau) \in S_\infty$$

respect to the variable x, that is, there are positive constants K_1, $\lambda > 0$, such that for any (x,t,τ), $(y,t,\tau) \in S_\infty$ have the estimates:

$$||f'_x(x,t,\tau)|| \leq K_1,$$

$$||f'_x(x,t,\tau) - f'_x(y,t,\tau)|| \leq \lambda |x-y|.$$

5. Uniformly in (x,t) there are limits:

$$F(x,t) \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T f(x,t,\tau)\,d\tau$$

and

$$F'_x(x,t) \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T f'_x(x,t,\tau)\,d\tau.$$

Along with the perturbed problem (3) consider the averaged problem $t \geq 0$

$$\begin{cases} \dfrac{dy}{dt} = Ay + F(y,t) \\ My(0) = \varphi. \end{cases} \tag{4}$$

6. Assume, that the problem (4) has bounded on the semi axis $t \geq 0$ solution $\overset{\circ}{y}(t)$ and at the same $F'_y(\overset{\circ}{y}(t),t) = 0$,

Remark 2 *If $\overset{\circ}{y}(t)$ is a stationary solution of the problem (4) and the nonlinear part of the problem is independent of time, then with the help of the transition from matrix A to matrix $A + F'_y(\overset{\circ}{y})$ requirement $F'_y(\overset{\circ}{y},t) = 0$ can be removed.*

Theorem 2 *Let the conditions 1 - 6 are satisfied. Then there exists $\omega_0 > 0$ such that in some $C^\mu([0;\infty))$ - neighborhood of vector function $\overset{\circ}{y}$ the problem (3) for $\omega > \omega_0$ has a unique limited solution x_ω and the following equality holds:*

$$\lim_{\omega\to\infty} ||x_\omega - \overset{\circ}{y}||_{C^\mu([0;\infty))} = 0.$$

Where $C^\mu([0;\infty))$ is Holder space on the semi axis $t \in [0;\infty)$, which is similar to space $C^\mu([0;1])$ (.1).

4 Scheme of the proof

For the sake of brevity, we do not prove here theorem 1, 2. We describe the scheme of the proofs. We need from problem (1), (3) to make the transition to the integral equations (see [3]), associate with them the operator equations in the appropriate spaces and apply abstract implicit function theorem (see [4]).Earlier this scheme was used [5], where the method of averaging is justified for abstract parabolic equations in the case of the Cauchy problem on a finite time interval and the problem of periodic solutions on the whole line.

5 Illustrative of the problem

5.1 Example 1

We consider the problem on the interval $t \in [0;1]$

$$\begin{cases} \dfrac{dx_1}{dt} = & -\pi x_2 + (x_1 - \exp(2t))^2 + 2\exp(2t) + \sin(\omega t) \\ \dfrac{dx_2}{dt} = & \pi x_1 + x_2^2 - \pi \exp(2t) + \sin^4(\omega t)\cos(\omega t) \\ x_1(0) = & 1 \\ x_2(1) = & 0 \end{cases}$$

Here the matrix $A = \begin{pmatrix} 0 & -\pi \\ \pi & 0 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $R = \begin{pmatrix} 0 & 1 \end{pmatrix}$.

We show that $\begin{vmatrix} L \\ R\exp A \end{vmatrix} \neq 0$,

$$\exp(At) = \begin{pmatrix} \cos \pi t & -\sin \pi t \\ \sin \pi t & \cos \pi t \end{pmatrix},$$

$$R\exp(A) = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \pi & -\sin \pi \\ \sin \pi & \cos \pi \end{pmatrix} = \begin{pmatrix} \sin \pi & \cos \pi \end{pmatrix},$$

$$\begin{vmatrix} L \\ R\exp A \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \sin \pi & \cos \pi \end{vmatrix} = \cos \pi \neq 0.$$

Averaged problem has the form

$$\begin{cases} \dfrac{dy_1}{dt} = & -\pi y_2 + (y_1 - \exp(2t))^2 + 2\exp(2t) \\ \dfrac{dy_2}{dt} = & \pi y_1 + y_2^2 - \pi \exp(2t) \\ y_1(0) = & 1 \\ y_2(1) = & 0, \end{cases}$$

so that $F(y(t)) = \begin{pmatrix} ((y_1(t)-\exp(2t))^2+2\exp(2t) \\ y_2^2(t)-\pi\exp(2t) \end{pmatrix}$. This problem has a solution $\overset{\circ}{y} = \begin{pmatrix} \exp(2t) \\ 0 \end{pmatrix}$. Obviously, $F'_y(\overset{\circ}{y}) = 0$.

So by theorem 1, there is a relatively unique solution x_ω of the perturbed problem and performed limiting equality

$$\lim_{\omega\to\infty} ||x_\omega - \overset{\circ}{y}||_{C^\mu([0;1])} = 0.$$

5.2 Example 2

We consider the problem on the semi axis $t \geq 0$

$$\begin{cases} \dfrac{dx_1}{dt} = -x_1 - x_2 + (x_1-1)^2 + 1 + \exp(-t) + \sin(\omega t) \\ \dfrac{dx_2}{dt} = 2x_1 - 4x_2 + (x_2 - \exp(-t))^2 - 2 + 3\exp(-t) + \sin^2(\omega t)\cos(\omega t) \\ x_1(0) = 1 \\ x_2(0) = 1 \end{cases}$$

Here the matrix $A = \begin{pmatrix} -1 & -1 \\ 2 & -4 \end{pmatrix}$ has eigenvalues

$$\lambda_1 = -3 \quad \lambda_2 = -2.$$

Matrix of the boundary condition has the form $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Number of components of the vector equals the number of rows matrix M and it has the form $\varphi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. In addition to the problem $\dfrac{dv}{dt} = Av$ with the boundary condition $Mv(0) = \varphi$ is satisfied Lopatinskii condition.

Averaged problem has the form

$$\begin{cases} \dfrac{dy_1}{dt} = -y_1 - y_2 + (y_1-1)^2 + 1 + \exp(-t) \\ \dfrac{dy_2}{dt} = 2y_1 - 4y_2 + (y_2 - \exp(-t))^2 - 2 + 3\exp(-t) \\ y_1(0) = 1 \\ y_2(0) = 1, \end{cases}$$

so that

$$F(y(t)) = \begin{pmatrix} (y_1(t)-1)^2 + 1 + \exp(-t) \\ (y_2(t) - \exp(-t))^2 - 2 + 3\exp(-t) \end{pmatrix}.$$

This problem has bounded on positive semi axis solution

$$\overset{\circ}{y} = \begin{pmatrix} 1 \\ \exp(-t) \end{pmatrix}.$$

Obviously, $F'_y(\overset{\circ}{y}) = 0$.

So by theorem 2, there is a relatively unique bounded solution x_ω of the perturbed problem and performed limiting equality

$$\lim_{\omega \to \infty} ||x_\omega - \overset{\circ}{y}||_{C^\mu([0;\infty))} = 0.$$

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